

The Steinhaus-Weil property: its converse, Solecki amenability and subcontinuity.

by

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Abstract.

The Steinhaus-Weil theorem that concerns us here is the ‘interior points’ property – that in a topological group a non-negligible set S has the identity as an interior point of $S^{-1}S$. There are various converses; the one that mainly concerns us is due to Simmons and Mospan. Here the group is locally compact, so we have a *Haar* reference measure η . The Simmons-Mospan theorem states that a (regular Borel) measure has such a Steinhaus-Weil property if and only if it is absolutely continuous with respect to the Haar measure. In Part I (Propositions 1-7, Theorems 1-4) we exploit the connection between the interior-points property and a selective form of infinitesimal invariance afforded by a certain family of *selective* reference measures σ , drawing on Solecki’s amenability at 1 (and using Fuller’s notion of subcontinuity). In Part II (Propositions 8,9, Theorems 5, 6) we develop a number of relatives of the Simmons-Mospan theorem. In Part III (Theorems 7, 8) we link this with topologies of Weil type.

Keywords. Steinhaus-Weil property, amenability at 1, subcontinuity, Simmons-Mospan theorem, Weil topology, interior points property, Haar measure, Lebesgue decomposition, left Haar null.

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Introduction

We begin by stating the Steinhaus-Weil Theorem in its simplest form (Steinhaus [Ste] for the line, Weil [Wei, §11, p. 50] for a locally compact group, Grosse-Erdmann [GroE]):

Theorem SW. *In a locally compact Polish group with (left) Haar measure η , for non-null Borel B , $B^{-1}B$ (and likewise BB^{-1}) contains a neighbourhood of the identity.*

The interior-point property of the measure-theoretically ‘non-negligible’ set B of the theorem is referred to as the *Steinhaus-Weil property*, which encompasses the category variant due to Piccard [Pic] and Pettis [Pet], cf. Cor. 2’ and Th. 7B. This important result has many ramifications; for example, it is basic to the theory of regular variation – see e.g. [BinGT, Th. 1.1.1].

We are also concerned here with its converse, the Simmons-Mospan theorem ([Sim, Th. 1], [Mos, Th. 7]), stated in its simplest form:

Theorem SM. *In a locally compact Polish group, a Borel measure has the Steinhaus-Weil property if and only if it is absolutely continuous with respect to Haar measure.*

The proof of this and related results emerges in §4 after due preparatory work; regarding the reduction of the non-separable to the separable case, see the closing remark of §4. The results below hinge on work of Solecki [Sol2] on amenability at 1 and the concept of subcontinuity (see §2 and below).

For G a topological group with metric d (briefly: metric group), denote by $\mathcal{M}(G)$ the family of Borel regular σ -finite measures on G with $\mathcal{P}(G) \subseteq \mathcal{M}(G)$ the probability measures ([Kec, §17E], [Par]), by $\mathcal{P}_{\text{fn}}(G)$ the larger family of finitely-additive regular probability measures (cf. [Bin]), and by $\mathcal{M}_{\text{sub}}(G)$ submeasures (monotone, finitely subadditive set functions μ with $\mu(\emptyset) = 0$). Here *regular* is taken to imply both *inner* regularity (inner approximation by compact subsets, also called the Radon property, as in [Bog1, II §7.1]), and *outer* regularity (outer approximation by open sets). We recall that a σ -finite Borel measure on a metric space is necessarily outer regular ([Bog1, II. Th. 7.1.7], [Kal, Lemma 1.34], cf. [Par, Th. II.1.2] albeit for a probability measure) and, when the metric space is complete, inner regular ([Bog1, II. Th. 7.1.7], cf. [Par, Ths. II.3.1 and 3.2]). For X a metric space, we denote by $\mathcal{K} = \mathcal{K}(X)$ the family of compact subsets of X (the *hyperspace* of X in §1, where we view it as a topological space under the Hausdorff metric, or the Vietoris topology). For $\mu \in \mathcal{M}(G)$ we write ${}_g\mu(\cdot) := \mu(g\cdot)$ and $\mu_g(\cdot) := \mu(\cdot g)$; $\mathcal{M}(\mu)$ denotes the μ -measurable sets of G and $\mathcal{M}_+(\mu)$ those of finite positive measure, and $\mathcal{K}_+(\mu) := \mathcal{K}(G) \cap \mathcal{M}_+(\mu)$. For G a Polish group, recall that $E \subseteq G$ is *universally measurable* ($E \in \mathcal{U}(G)$) if E is measurable with respect to every measure $\mu \in \mathcal{P}(G)$ – for background, see e.g. [Kec, §21D], cf. [Fre, 434D, 432], [Sho]; these form a σ -algebra. Examples are analytic subsets (see e.g. [Rog, Part 1 §2.9], or [Kec, Th. 21.10], [Fre, 434Dc]) and the σ -algebra that they generate. Beyond these are the *provably* Δ_2^1 sets of [FenN].

Recall that E is *left Haar null*, $E \in \mathcal{HN}$, as in Solecki [Sol1,2,3] (following [Chr1,2]) if there are $B \in \mathcal{U}(G)$ covering E and $\mu \in \mathcal{P}(G)$ with

$$\mu(gB) = 0 \quad (g \in G).$$

(The terminal brackets here and below indicate *universal quantification* over the free variable.) So if $B \in \mathcal{U}(G)$ is not left Haar null, then for each $\mu \in \mathcal{P}(G)$ there is compact $K = K_\mu \subseteq B$ and $g \in G$ with

$${}_g\mu(K) > 0.$$

The question then arises whether there is also $\delta > 0$ with ${}_g\mu(Kt) > 0$ for *all* $t \in B_\delta$, for $B_\delta = B_\delta(1_G)$ the open δ -ball centered at 1_G : a *right-sided* property *complementing* left-sided nullity. If this is the case for some μ , then (see Corollary 2' in §2) $1_G \in \text{int}(K^{-1}K) \subseteq \text{int}(E^{-1}E)$; indeed, one has

$$K \cap Kt \in \mathcal{M}_+({}_g\mu) \quad (t \in B_\delta), \quad (*M)$$

and this implies (see Lemma 1, §2):

$$B_\delta \subseteq \text{int}(K^{-1}K) \subseteq \text{int}(E^{-1}E). \quad (**)$$

As this clearly forces local-compactness of G (see Lemma 1 below), for the more general context we weaken the ‘complementing right-sided property’ to hold only *selectively*: on a subset of B_δ of the form

$$\{z \in B_\delta : |\mu(Kz) - \mu(K)| < \varepsilon\}$$

(cf. $B_\delta^\Delta(\mu)$ in §2). This turns attention to refining the topology of G via the metrics

$$d_K(x, y) = d(x, y) + |\mu(Kx) - \mu(Ky)|,$$

determined in Theorem 5 below, aiming at an interior-points property relative to the finer topology. See (in addition to Theorem SW above) Kemperman [Kem] (cf. [Kuc, Lemma 3.7.2], [BinO1, Th. K], [BinO5, Th. 1(iv)]).

Our focus is the close relation between the measure-theoretic *Steinhaus-Weil-like* property (*M) and its category version

$$K \cap Kt \in \mathcal{B}_+(\mathcal{T}_G), \quad (*B)$$

where the latter term refers to non-meagre *Baire* sets (= with the Baire property) in the ambient topology $\mathcal{T}_G = \mathcal{T}_d$ of G , conveniently taken to be

generated by a *left*-invariant metric $d = d_L^G$ with associated group-norm (§5) $\|x\| := d(x, 1_G) = d(tx, t)$ (so that $B_\delta(t) = tB_\delta$ – see Prop. 1).

In Part I below (§1,2, Props. 1-7, Th. 1-4: Subcontinuity Theorem, Aggregation Theorem, Shift-compactness Theorem, Strong Subcontinuity Theorem) in the context of a metric or Polish group G we study continuity properties of the maps $m_K : t \mapsto \mu(Kt)$ in the light of theorems of Solecki [Sol2] and of Theorem SM above and related results. The key here is Fuller’s notion of subcontinuity, as applied to the function $m_K(t)$ at $t = 1_G$. This yields a fruitful interpretation of Solecki’s notion of *amenability at 1_G* via *selective subcontinuity* and linking it to *shift-compactness* (see Th. 3 below; the term is borrowed from [Par, III.2]). Since commutative Polish groups are amenable at 1 [Sol2, Th. 1(ii)], this widens the field of applicability of shift-compactness to non-Haar-null subsets of these, as in [BinO6]. In Part II (§3,4, Props. 8-9, Theorems 5,6: Disaggregation Theorem, Generalized Simmons Theorem) we study measure discontinuity and extend results of Simmons beyond his locally compact context by reference to measures offering selective subcontinuity. In Part III (§6, 7, Props. 10, 11, Theorems 7, 7M, 7B, 8) we study the converse problem of the *Weil topologies* generated by the pseudo-norms, defined for $E \in \mathcal{M}_+(G)$ by:

$$\|t\|_\mu^E := \mu(tE \triangle E) \quad (t \in G).$$

As here, we study EE^{-1} when following Simmons and Weil, and $E^{-1}E$ when following Solecki, for ease of reference – duly identified.

For further detail and background, see the Appendix.

Part I: Subcontinuity and Solecki’s amenability

1 Measure under translation – preliminaries

We begin with a form of the ‘telescope’ or ‘tube’ lemma (cf. [Mun, Lemma 5.8]), applied in §2. Our usage of upper semicontinuity in relation to set-valued maps follows [Rog], cf. [Bor].

Proposition 1. *For a metric group G and compact $K \subseteq G$, the map $t \mapsto Kt$ is upper semicontinuous; in particular, for $\mu \in \mathcal{M}(G)$,*

$$m_K : t \mapsto \mu(Kt)$$

is upper semicontinuous, hence μ -measurable. In particular, if $m_K(t) = 0$, then m is continuous at t .

Proof. For K compact and $V \supseteq K$ open, pick for each $k \in K$ a $\delta(k) > 0$ with $kB_{2\delta(k)} \subseteq V$. By compactness, there are k_1, \dots, k_n with $K \subseteq \bigcup_j k_j B_{\delta(k_j)} \subseteq V$; then for $\delta := \min_j \delta(k_j) > 0$

$$Kt \subseteq \bigcup_j k_j B_{\delta(k_j)} t \subseteq \bigcup_j k_j B_{2\delta(k_j)} \subseteq V \quad (||t|| < \delta).$$

To prove upper semicontinuity of m_K , fix $t \in G$. For $\varepsilon > 0$, as Kt is compact, choose by outer regularity an open $U \supseteq Kt$ with $\mu(U) < \mu(Kt) + \varepsilon$; as before, there is an open ball B_δ at 1_G with $KtB_\delta \subseteq U$, and then $\mu(KtB_\delta) \leq \mu(U) < \mu(Kt) + \varepsilon$. The final assertion follows from positivity of m_K . \square

We continue with an analogue. The result is folklore, cf. [BeeV, Th. 3.2(i)]; it comes close to matters touched on in [Ost1, §3]. Here and below the vertical section of a set A is denoted $A_x := \{y : (x, y) \in A\}$.

Proposition 2 (Sectional upper semicontinuity). *For a metric group G , compact $F \subseteq G$ and compact $K \subseteq G^2$, the map*

$$x \mapsto K_x \quad (x \in F)$$

is upper semicontinuous.

Proof. For $V \subseteq G$ open with $K_x \subseteq V$, suppose for $x_n \in F$ with $x_n \rightarrow x$ that $(x_n, y_n) \in K \setminus (G \times V)$. By compactness of K , we may suppose w.l.o.g. that $y_n \rightarrow y$. Then $(x, y) \in K \setminus (G \times V)$, and so $(x, y) \in \{x\} \times K_x$ and $y \notin V$; but $y \in K_x \subseteq V$, a contradiction. \square

From Prop. 2 on *upper* semicontinuity, we obtain information about $m_K : t \mapsto \mu(Kt)$ below. (At no extra cost, this also yields a direct proof of the Fubini Theorem for null sets, Theorem FN below, in the manner of van Douwen [vDo]: see the Appendix.) This links with *lower* semicontinuity. By a theorem of Fort, the ε -continuity points (defined in terms of the Hausdorff metric: see [For]) of an upper semicontinuous compact-valued mapping of a metric space into a totally bounded metric space form a dense open set, implying in a real-valued context such as here *continuity* on a co-meagre set. We return to this shortly in Theorem LB below.

Proposition 3 (Sectional upper semicontinuity under a measure).

For a metric group G , compact $F \subseteq G$ and compact $K \subseteq G^2$, and $\mu \in \mathcal{M}(G)$, the map

$$m : x \mapsto \mu(K_x) \quad (x \in F = \text{proj}_1 K)$$

is upper semicontinuous, and so Borel.

Proof. Fix $x \in F$. Let $\varepsilon > 0$. By outer regularity, take V open in G with $K_x \subseteq V$ and $\mu(V) < \mu(K_x) + \varepsilon$. By Prop. 2 $x \mapsto \{x\} \times K_x$ is upper semicontinuous on F ; so for some open neighbourhood U of x

$$K \cap (U \times G) \subseteq K \cap (U \times V).$$

So for $y \in F \cap U$

$$K_y \subseteq V,$$

and so $\mu(K_y) \leq \mu(V) < \mu(K_x) + \varepsilon$, proving the first assertion. The second assertion follows since

$$m^{-1}(a, b) = \bigcap_{n \in \mathbb{N}} m^{-1}[0, b) \setminus m^{-1}[0, a + 1/n). \quad \square$$

For further results on Borel-measurability of regular Borel measures see [BeeV, Th. 2.2] (there termed ‘Radon measures’).

We will need the following result in Parts II and III (see Lemma 2, §4, and Th. 7, §6), preferable to the usual Fubini Theorem as using qualitative rather than quantitative measure theory (like the Kuratowski-Ulam Theorem [FreNR]). Interestingly, it may be proved by mimicking the proof of Prop. 1 above, yielding a simplification to that by Eric van Doven [vDo], itself a simplification of that in [Oxt2, Ch. 14]: for the proof (omitted here), see the Appendix.

Theorem FN (Fubini theorem for null sets). *For a metric group G and $A \subseteq G^2$ measurable under $\mu \times \nu$, with $\mu, \nu \in \mathcal{M}(G)$: if the ‘exceptional set’ of points x for which the vertical section A_x is ν -non-null is itself μ -null, then A is $\mu \times \nu$ -null.*

We close this section with a study of the continuity properties of the map $m_K : t \mapsto \mu(Kt)$ for compact K , extending Prop. 3.

Corollary 1 (Fort [For]). *In Proposition 2, $t \mapsto \mu(Kt)$ is lower semi-continuous (so also continuous) on a co-meagre set.*

We can improve on the preceding result by recourse to a natural generalization, for our compact sectional context, of the classical continuity theorems of Luzin [Hal, §55] and Baire [Oxt2, Th. 8.1]. Below for a *compact* metric space X , we denote by $\mathcal{K}(X)$ the *hyperspace of X* , the space of compact subsets of X under the Hausdorff metric, or Vietoris topology; here this is also a compact space ([Eng, 2.7.28], [Kec, Th. 4.25], [Mic]). Then (LB for ‘Luzin-Baire’):

Theorem LB. *For G a metric group and compact $K \subseteq G^2$, the map $\kappa : G \rightarrow \mathcal{K}(G) : x \mapsto K_x$ is Borel-measurable, and so*

(i) *κ is continuous relative to a co-meagre set.*

For $\mu \in \mathcal{P}(G)$:

(ii) *for each $\varepsilon > 0$ there is a Borel set S_ε with $\mu(G \setminus S_\varepsilon) < \varepsilon$ such that $x \mapsto K_x$ is continuous on S_ε ; equivalently:*

(ii)' *there is an increasing sequence of Borel sets S_n with union μ -almost all of G such that $x \mapsto K_x$ is continuous on each S_n .*

Proof. For U, V open, the set $\{x : K_x \subseteq U \text{ and } K_x \cap V \neq \emptyset\}$ is σ -compact; indeed, by Prop. 2, $\{x : K_x \subseteq U\}$ is open, whereas for $F := \text{proj}_1(K)$, a compact set

$$\{x : K_x \cap U \neq \emptyset\} = \text{proj}_1[(F \times U) \cap K],$$

which is the projection of a σ -compact set. Then (ii) follows from Luzin’s theorem ([Hal, §55]), and for (ii)' see [BinO1, p. 142]. Its extension to a regular (i.a. \mathcal{G} -outer regular) σ -finite measure may be made via Egoroff’s Theorem [Hal, §21 Th. A] – see [Zak]. \square

A first corollary is the following result on the continuity of the map

$$x \mapsto \|x\|_E^\mu = \mu(xE \triangle E),$$

for *measurable* E , by compact approximation. Below, the sets C_x associated with points x should be interpreted as neighbourhoods of x in the spirit of a Hashimoto ideal topology for the ideal of μ -null sets, for which see [LukMZ], or [BinO5]. This mimicks Weil’s *proof* of the ‘fragmentation lemma’ in [Hal, Ch. XII §62 Th. A] (cf. [Wei, Ch. VII, §31]).

Proposition 4 (Almost everywhere continuity). *For a metric group G , $\delta > 0$, $\mu \in \mathcal{P}(G)$, $E \in \mathcal{M}_+(\mu)$, and compact $F \in \mathcal{M}_+(\mu)$:*

there is $C \subseteq F$ with $\mu(F \setminus C) < \delta$ such that for any $\varepsilon > 0$ and each $x \in C$ there is a μ -non-null measurable $C_x \subseteq C$ containing x with

$$|\mu(xE \triangle E) - \mu(yE \triangle E)| < \varepsilon \quad (y \in C_x).$$

In particular, there is an increasing family of compact sets C_n with union μ -almost all of G satisfying the above with C_n for C .

Proof. Fix $E \subseteq G$ measurable and for F compact with $\mu(F) > 0$, put

$$\begin{aligned} H &:= \bigcup_{x \in F} \{x\} \times (E \triangle xE) = \bigcup_{x \in F} \{x\} \times ((xE \setminus E) \cup (E \setminus xE)) \\ &= \bigcup_{x \in F} (\{x\} \times xE) \setminus \bigcup_{x \in F} (\{x\} \times E) \cup \bigcup_{x \in F} (\{x\} \times E) \setminus \bigcup_{x \in F} \{x\} \times (xE), \end{aligned}$$

which is measurable. So $F \subseteq \text{proj}_1(H)$ and for $x \in F$

$$H_x := xE \triangle E.$$

We will work inductively, taking successively smaller values of ε . Fix $\varepsilon > 0$ with $\varepsilon < \mu(F)$ and choose K a finite union of compact rectangles with

$$(\mu \times \mu)(H \triangle K) < \varepsilon^2.$$

So

$$K = \bigcup_{j \leq n} (F^j \times K^j),$$

say. Let $S = S_\varepsilon := \{x \in F : \mu((H \triangle K)_x) \geq \varepsilon\}$. Then $\mu(S) \leq \varepsilon$; otherwise

$$\varepsilon^2 > (\mu \times \mu)(H \triangle K) \geq \varepsilon \mu(S) \geq \varepsilon^2,$$

a contradiction. So

$$\mu(F \setminus S) > \mu(F) - \varepsilon.$$

So we may choose a compact set $C = C_\varepsilon \subseteq F \setminus S_\varepsilon$ with $\mu(F \setminus C) < \varepsilon$ and

$$\mu(H_x \triangle K_x) \leq \varepsilon \quad (x \in C_\varepsilon).$$

But $K_x \in \{K^1, \dots, K^n\}$, so

$$C_\varepsilon = \bigcup_{j \leq n} C_\varepsilon^j, \quad \text{where } C_\varepsilon^j := \{x \in C_\varepsilon : \mu(H_x \triangle K^j) \leq \varepsilon\}.$$

Now for $x, y \in C_\varepsilon^j$

$$\mu(H_x \triangle H_y) \leq \mu(H_x \triangle K^j) + \mu(K^j \triangle H_y) \leq 2\varepsilon,$$

and w.l.o.g. the sets C_ε^j may be assumed compact.

Now fix $\delta > 0$ and repeat the construction inductively, taking in turn $\varepsilon = \varepsilon_n = 2^{-n}\mu(F)\delta$, to obtain a sequence of compact sets $F = C^0 \supseteq C = C^1 \supseteq C^2 \supseteq C^3 \dots$ with $C^n = C_{\varepsilon_n}$ as above, and $\mu(C^n \setminus C^{n+1}) < \varepsilon_n$. These have non-null intersection C_0 :

$$\mu(F \setminus C_0) < \sum_{n \geq 1} 2^{-n}\mu(F)\delta = \mu(F)\delta.$$

Now for each $\varepsilon > 0$ there is n with $\varepsilon_n < \varepsilon$. So for $x \in C_0 \subseteq C^n = C_{\varepsilon_n} = \bigcup_{j \leq n} C_{\varepsilon_n}^j$

$$\mu(H_x \triangle H_y) < \mu(H_x \triangle H_y) + 2\varepsilon_n < \mu(H_x) + 2\varepsilon \quad (x, y \in C_{\varepsilon_n}^j \cap C_0).$$

So

$$|\mu(H_x) - \mu(H_y)| < 4\varepsilon \quad (x, y \in C_{\varepsilon_n}^j \cap C_0). \quad \square$$

A proof similar to but simpler than that above (omitted here – see the Appendix) improves Prop. 1:

Proposition 5 (Almost everywhere upper semicontinuity). *For a metric group G , $\delta > 0$, $\mu \in \mathcal{P}(G)$, E measurable with $\mu(E) > 0$, and F compact with $\mu(F) > 0$:*

there is $C \subseteq F$ with $\mu(F \setminus C) < \delta$ such that for any $\varepsilon > 0$ each $x \in C$ has a neighbourhood U_x with

$$\mu(yE) < \mu(xE) + \varepsilon \quad (y \in C \cap U_x).$$

In particular, there are disjoint compact sets C with union μ -almost all of G for which this holds.

2 Subcontinuity of measures

Proposition 1 above, on upper semicontinuity, motivates the following definitions, the key one being an adaptation of *subcontinuity* (of functions) due to Fuller [Ful] (for which see Remark 4 below) to the context of measures. We

focus on the *right-sided* version of the concept. Subcontinuity is a natural auxiliary in the quest for fuller forms of continuity: as one instance, see [Bou] for the step from separate to joint continuity; as another, classic instance, note that a subcontinuous set-valued map with closed graph (yet another relative of upper semicontinuity) is continuous – see [HolN] for an extensive bibliography. Here its relevance to the Steinhaus-Weil Theorem (which seems to be new here) yields Theorems 1 and 3, linking *amenability at 1* with *shift-compactness* for which see Theorem 3 below (the latter term is borrowed from [Par, III.2]).

Definition. For $\mu \in \mathcal{P}_{\text{fin}}(G)$, and (compact) $K \in \mathcal{K}(G)$, noting that $\mu_\delta(K) := \inf\{\mu(Kt) : t \in B_\delta\}$ is weakly decreasing in δ , put

$$\mu_-(K) := \sup_{\delta > 0} \inf\{\mu(Kt) : t \in B_\delta\},$$

and, for $\mathbf{t} = \{t_n\}$ a *null sequence*, i.e. with $t_n \rightarrow 1_G$,

$$\mu_-^{\mathbf{t}}(K) := \liminf_{n \rightarrow \infty} \mu(Kt_n).$$

Then

$$0 \leq \mu_-(K) \leq \mu(K) = \inf_{\delta > 0} \sup\{\mu(Kt) : t \in B_\delta\},$$

by Proposition 1. We say that a null sequence \mathbf{t} is *non-trivial* if $t_n \neq 1_G$ infinitely often. Define (with the quantification convention of the Introduction) as follows:

- (i) μ is *translation-continuous* (*‘continuous’*) if $\mu(K) = \mu_-(K)$ ($K \in \mathcal{K}(G)$);
- (ii) μ is *maximally discontinuous* at $K \in \mathcal{K}(G)$ if $0 = \mu_-(K) < \mu(K)$;
- (iii) μ is *subcontinuous* if $0 < \mu_-(K) \leq \mu(K)$ ($K \in \mathcal{K}_+(\mu)$);
- (iv) μ is (*selectively*) *subcontinuous at* $K \in \mathcal{K}_+(\mu)$ *along* \mathbf{t} if $\mu_-^{\mathbf{t}}(K) > 0$.

Remarks. 1. $m_K(\cdot)$ is *continuous* if μ is continuous, since $m_K(st) = m_{Ks}(t)$ and Ks is compact whenever K is compact; for directional continuity of measures in linear spaces see [Bog2, §3.1].

2. For G locally compact (i) holds for μ the left Haar measure η , and also for $\mu \ll \eta$ (absolutely continuous w.r.t. to η).

3. A measure μ singular w.r.t. Haar measure is maximally discontinuous for its support: this is at the heart of the analysis offered by Simmons (and independently, much later by Mospan) – see Corollary 2' below.

4. *Subcontinuity*, in the sense of [Ful], of a map $f : G \rightarrow (0, \infty)$ requires that, for every $t_n \rightarrow t \in G$, there is a subsequence $t_{m(n)}$ with $f(t_{m(n)})$ convergent in the range (i.e. to a positive value). The distinguished role of null sequences emerges below in the *Subcontinuity Theorem* (Theorem 1). Null sequences should be viewed here as selecting stepwise (or even pathwise, under local connectedness, as suggested by Tomasz Natkaniec) ‘asymptotic directions’ justifying the phrase ‘*along t*’ in (iv) above, and allowing (iv) to be interpreted as a *selective subcontinuity* in ‘direction’ \mathbf{t} . The analogous selective concept in a linear space is ‘along a vector’ as in [Bog2, §3.1].
5. *Selective versus uniform subcontinuity*. Definition (iii) is equivalent to demanding for $K \in \mathcal{K}(G) \cap \mathcal{M}_+(\mu)$ that *any* null sequence $\mathbf{t} = \{t_n\}$ have a subsequence $\mu(Kt_{m(n)})$ bounded away from 0; then (iii) may be viewed as demanding ‘uniform subcontinuity’: selective subcontinuity along *each* \mathbf{t} for all $K \in \mathcal{K}(G) \cap \mathcal{M}_+(\mu)$.
6. *Left- versus right-sided versions*. Writing $\tilde{\mu}(E) := \mu(E^{-1})$ for the *inverse measure* captures versions associated with right-sided translation such as $\tilde{\mu}_-$ and

$$\tilde{\mu}_-^{\mathbf{t}}(K) := \liminf_{n \rightarrow \infty} \mu(t_n K).$$

Definition. We will say that μ is *symmetric* if $\mu = \tilde{\mu}$; then B is null iff B^{-1} is null.

In Lemma 1 below it suffices for μ to be a bounded, regular submeasure which is *supermodular*:

$$\mu(E \cup F) \geq \mu(E) + \mu(F) - \mu(E \cap F);$$

recall, however, from [Bog1, 1.12.37] the opportunity to replace, for any $K \in \mathcal{K}(G)$, a supermodular submeasure μ by a dominating $\mu' \in \mathcal{M}_{\text{fin}}(G)$, i.e. with $\mu'(K) \geq \mu(K)$.

For $K \in \mathcal{K}(G) \cap \mathcal{M}_+(\mu)$ and $\delta, \Delta > 0$, put

$$B_\delta^\Delta = B_\delta^{K, \Delta}(\mu) := \{z \in B_\delta : \mu(Kz) > \Delta\},$$

which is monotonic in $\Delta : B_\delta^\Delta \subseteq B_\delta^{\Delta'}$ for $0 < \Delta' \leq \Delta$. Note that $1_G \in B_\delta^\Delta$ for $0 < \Delta < \mu(K)$.

Lemma 1. *Let $\mu \in \mathcal{P}_{\text{fin}}(G)$ for G a metric group. For $K \in \mathcal{K}_+(\mu)$, if $\mu_-^{\mathbf{t}}(K) > 0$ for some non-trivial null sequence \mathbf{t} , then there are $\delta > 0$ and $0 < \Delta < \mu_-^{\mathbf{t}}(K)$ with $t_n \in B_\delta^\Delta$ for all large enough n and*

$$\Delta \leq \mu(K \cap Kt) \quad (t \in B_\delta^\Delta),$$

so that

$$K \cap Kt \in \mathcal{M}_+(\mu) \quad (t \in B_\delta^\Delta). \quad (*)$$

In particular,

$$K \cap Kt \neq \emptyset \quad (t \in B_\delta^\Delta),$$

or, equivalently,

$$B_\delta^\Delta \subseteq K^{-1}K, \quad (**)$$

so that B_δ^Δ has compact closure.

A fortiori, if $\mu_-(K) > 0$, then $\delta, \Delta > 0$ may be chosen with $\Delta < \mu_-(K)$ and $B_\delta \subseteq B_\delta^\Delta$ so that $(*)$ and $(**)$ hold with B_δ replacing B_δ^Δ , and in particular G is locally compact.

Proof. For the first part take $\Delta := \mu_-^\mathbf{t}(K)/4$. Then, for $t \in B_\delta^\Delta$ with $\delta > 0$ arbitrary, $\mu(Kt) > 2\Delta$, and so, since $t_n \in B_\delta$ for all large enough n , also $t_n \in B_\delta^\Delta$ for all large enough n , so $B_\delta^\Delta(K) \setminus \{1_G\}$ is non-empty for \mathbf{t} non-trivial.

Put $H_t := K \cap Kt \subseteq K$. By outer regularity of μ , choose U open with $K \subseteq U$ and $\mu(U) < \mu(K) + \Delta$. By upper semicontinuity of $t \mapsto Kt$, w.l.o.g. $KB_\delta \subseteq U$ for some $\delta > 0$. For $t \in B_\delta^\Delta$, by finite additivity of μ , since $2\Delta < \mu(Kt)$

$$\begin{aligned} 2\Delta + \mu(K) - \mu(H_t) &\leq \mu(Kt) + \mu(K) - \mu(H_t) = \mu(Kt \cup K) \\ &\leq \mu(U) \leq \mu(K) + \Delta. \end{aligned}$$

Comparing extreme ends of this chain of inequalities gives

$$0 < \Delta \leq \mu(H_t) \quad (t \in B_\delta^\Delta).$$

For $t \in B_\delta^\Delta$, as $K \cap Kt \in \mathcal{M}_+(\mu)$, take $s \in K \cap Kt \neq \emptyset$; then $s = kt$ for some $k \in K$, so $t = k^{-1}s \in K^{-1}K$. Conversely, $t \in B_\delta^\Delta \subseteq K^{-1}K$ yields $t = k^{-1}k'$ for some $k, k' \in K$; then $k' = kt \in K \cap Kt$.

By the compactness of $K^{-1}K$, B_δ^Δ has compact closure.

As for the final assertions, if $\mu_-(K) > 0$, take $\Delta := \mu_-(K)/2$. Then $\inf\{\mu(Kt) : t \in B_\delta\} > \Delta$ for all small enough $\delta > 0$, and so in particular $\mu(Kt) > \Delta$ for $t \in B_\delta$, i.e. $B_\delta \subseteq B_\delta^\Delta$. So the argument above applies for such $\delta > 0$ with B_δ in lieu of B_δ^Δ , just as before. Here the compactness of $K^{-1}K$ now implies local compactness of G itself. \square

As an immediate and useful corollary, we have

Lemma 1'. For $\mu \in \mathcal{P}_{\text{fin}}(G)$, with G a metric group, any null sequence \mathbf{t} and any $K \in \mathcal{K}(G)$: if $\mu_{-}^{\mathbf{t}}(K) > 0$, then there is $m \in \mathbb{N}$ with

$$0 < \mu_{-}^{\mathbf{t}}(K)/4 < \mu(K \cap Kt_n) \quad (n > m). \quad (*')$$

In particular,

$$t_n \in K^{-1}K \quad (n > m). \quad (**')$$

Proof. Apply Lemma 1 to obtain $\Delta, \delta > 0$; for $t \in B_{\delta}^{\Delta}$, $\mu(Kt) > \Delta$, so as above $t_n \in B_{\delta}^{\Delta}$ for all large enough n . \square

This permits a connection with left Haar null sets; recall that a group G is *amenable at 1* [Sol2] (see below for the origin of this term) if given $\boldsymbol{\mu} := \{\mu_n\} \in \mathcal{P}(G)$ for $n \in \mathbb{N}$ with $1_G \in \text{supp}(\mu_n)$ there are σ and σ_n in $\mathcal{P}(G)$ with $\sigma_n \ll \mu_n$ for all n and

$$\sigma_n * \sigma(K) \rightarrow \sigma(K) \quad (K \in \mathcal{K}(G)).$$

We refer to σ (or $\sigma(\boldsymbol{\mu})$ if context requires) as the *Solecki measure* and to the measures σ_n , if needed, as the *associated measures* (corresponding to the sequence μ_n).

Solecki explains ([Sol2, end of §2]) the use of the term ‘amenability at 1’ as a localization (via the restriction of supports to contain 1_G) of a *Reiter-like condition* [Pat, Prop. 0.4] which characterizes amenability: for $\mu \in \mathcal{P}(G)$ and $\varepsilon > 0$, there is $\nu \in \mathcal{P}(G)$ with

$$|\nu * \mu(K) - \nu(K)| < \varepsilon \quad (K \in \mathcal{K}(G)).$$

Lemma 1 and the next several results disaggregate Solecki’s Interior-point Theorem [Sol2, Th 1(ii)] (Corollary 2 below), shedding more light on it and in particular connecting it to shift-compactness (Theorem 3 below). Indeed, we see that interior-point theorem itself as an ‘aggregation’ phenomenon. Theorem 5 of Part II below identifies subgroups with a ‘disaggregation’ topology, refining \mathcal{T}_G by using sets of the form $B_{\delta}^{K\Delta}(\sigma)$, the measures σ being provided in our first result:

Theorem 1 (Subcontinuity Theorem, after Solecki [Sol2, Th. 1(ii)]). For G Polish and amenable at 1_G and \mathbf{t} a null sequence, there is $\sigma = \sigma(\mathbf{t}) \in$

$\mathcal{P}(G)$ such that for each $K \in \mathcal{K}(G)$ with $\sigma(K) > 0$ there is a subsequence $\mathbf{s} = \mathbf{s}(K) := \{t_{m(n)}\}$ with

$$\lim_n \sigma(Kt_{m(n)}) = \nu(K) \quad (n \in \mathbb{N}), \text{ so } \quad \sigma_{\mathbf{s}}^{\mathbf{s}}(K) > 0.$$

Proof. For $\mathbf{t} = \{t_n\}$ null, put $\mu_n := 2^{n-1} \sum_{m \geq n} 2^{-m} \delta_{t_m^{-1}} \in \mathcal{P}(G)$; then $1_G \in \text{supp}(\mu_n) \supseteq \{t_m^{-1} : m \geq n\}$. By definition of amenability at 1_G , in $\mathcal{P}(G)$ there are σ and $\sigma_n \ll \mu_n$, with $\sigma_n * \sigma(K) \rightarrow \sigma(K)$ for all $K \in \mathcal{K}(G)$. Choose $\alpha_{mn} \geq 0$ with $\sum_{m \geq n} \alpha_{mn} = 1$ ($n \in \mathbb{N}$) and with $\sigma_n := \sum_{m \geq n} \alpha_{mn} \delta_{t_m^{-1}}$.

Fix $K \in \mathcal{K}(G)$ with $\sigma(K) > 0$ and θ with $0 < \theta < 1$. As K is compact, $\sigma_n * \sigma(K) \rightarrow \sigma(K)$; then w.l.o.g.

$$\sigma_n * \sigma(K) > \theta \sigma(K) \quad (n \in \mathbb{N}).$$

Then for each n

$$\sup\{\sigma(Kt_m) : m \geq n\} \cdot \sum_{m \geq n} \alpha_{mn} \geq \sum_{m \geq n} \alpha_{mn} \sigma(Kt_m) > \theta \sigma(K).$$

So for each n there is $m = m(\theta) \geq n$ with

$$\sigma(Kt_m) > \theta \sigma(K).$$

Now choose $m(n) \geq n$ inductively so that $\sigma(Kt_{m(n)}) > (1 - 2^{-n})\sigma(K)$; then, by Proposition 1, $\lim_n \sigma(Kt_{m(n)}) = \sigma(K)$: σ is subcontinuous along $\mathbf{s} := \{t_{m(n)}\}$ on K . \square

We are now able to deduce Solecki's interior-point theorem in a slightly stronger form, which asserts that the sets B_δ^Δ reconstruct the open sets of G using the compact subsets of a 'non-negligible set', as follows.

Theorem 2 (Aggregation Theorem). *For G Polish and amenable at 1_G , if $E \in \mathcal{U}(G)$ is not left Haar null – then, setting*

$$\hat{E} := \bigcup_{\delta, \Delta > 0, g \in G, \mathbf{t}} \{B_\delta^{gK, \Delta}(\nu(\mathbf{t})) : K \subseteq E, K \in \mathcal{K}_+(\nu(\mathbf{t})), \Delta < \nu(gK)\},$$

$$1_G \in \text{int}(\hat{E}) \subseteq \hat{E} \subseteq E^{-1}E.$$

In particular, for E open, $1_G \in \text{int}(\hat{E})$.

Proof. Suppose otherwise; then for each n there is

$$t_n \in B_{1/n} \setminus \hat{E}.$$

Consider $\sigma = \sigma(\mathbf{t})$. As E is not left Haar null, there is g with $\sigma(gE) > 0$. Choose compact $K \subseteq gE$ with $\sigma(K) > 0$. Then with $h := g^{-1}$ and $H := hK \subseteq E$, $\sigma(K) = \sigma(gH) = \sigma^{\mathbf{s}}(gH) > 0$ for some subsequence $\mathbf{s} = \{t_{m(m)}\}$. So with $\Delta := \sigma(gH)/4$ for some $\delta > 0$

$$B_\delta^{gH\Delta}(\sigma(\mathbf{t})) \subseteq (gH)^{-1}gH = H^{-1}H \subseteq E^{-1}E.$$

Choose n with $n > 1/\delta$. Then $t_n \in B_\delta$ for all $m > n$; so for infinitely many k

$$t_{m(k)} \in B_\delta^{gH\Delta}(\sigma(\mathbf{t})) \subseteq \hat{E},$$

a contradiction. As for the last assertion, for E open, D countable and dense, $G \subseteq \bigcup_{d \in D} dE$, so for $\mu \in \mathcal{P}(G)$ (in particular for σ) $\mu(dE) > 0$ for some $d \in D$, and so E is not left Haar null. \square

The immediate consequence is

Corollary 2 (Solecki's Interior-Point Theorem [Sol2, Th 1(ii)]). *For G Polish and amenable at 1_G , if $E \in \mathcal{U}(G)$ is not left Haar null, then $1_G \in \text{int}(E^{-1}E)$.*

Corollary 2'. *For G a Polish group, if $B \in \mathcal{U}(G)$ is not left Haar null and is in $\mathcal{M}_+(\mu)$ for some subcontinuous $\mu \in \mathcal{P}_{\text{fin}}(G)$, then $(**)$ holds for some $\delta > 0$.*

*In particular, $(**)$ holds in a locally compact group G , for any Baire non-meagre set E .*

Proof. The first assertion is immediate from Lemma 1. As for the second, for a non-meagre Baire set E , if \tilde{E} is the quasi-interior and $K \subseteq \tilde{E}$ is compact with non-empty interior, then $\eta(K) > 0$. Since η is subcontinuous, there is $\delta > 0$ with

$$Kt \cap K \neq \emptyset \quad (||t|| < \delta).$$

A fortiori,

$$\tilde{E}t \cap \tilde{E} \neq \emptyset \quad (||t|| < \delta);$$

then $U := (Et)^\sim \cap \tilde{E} \neq \emptyset$, since $(Et)^\sim = \tilde{E}t$ (the Nikodym property of the *usual* topology of G). So since U is open and non-meagre, also $Et \cap E \neq \emptyset$, and so again (**). \square

The next result establishes the embeddability by (left-sided) translation of an appropriate subsequence of a given *null* sequence into a given target set that (like-sidedly) is non-left-Haar null. This property of embedding into a *non-negligible* set, first studied in respect of category and measure negligibility on \mathbb{R} by Kestelman and much later independently by Borwein and Ditor and thereafter also by other authors, mostly for combinatorial challenges, has emerged as an important general unifying principle, termed shift-compactness, applicable in a much wider context embracing metric groups G under various topologies refining \mathcal{T}_G and so defining various notions of negligibility: for the background here see [BinO2,3], [MilO]. Its consequences include various uniform-boundedness theorems as well as the Effros and the Open Mapping Theorem. Here we establish the said property, announced in [MilO], in relation to the ideal \mathcal{HN} of left Haar null sets. (It is a σ -ideal for Polish G in the presence of amenability at 1 [Sol2, Th 1(i)].) This leaves open the ‘converse question’ of a refinement topology for which \mathcal{HN} is the associated notion of negligibility.

Theorem 3 (Shift-compactness Theorem for \mathcal{HN}). *For G Polish and amenable at 1_G , if $E \in \mathcal{U}(G)$ is not left Haar null and z_n is null, then there are $s \in E$ and an infinite $\mathbb{M} \subseteq \mathbb{N}$ with*

$$\{sz_m : m \in \mathbb{M}\} \subseteq E.$$

Indeed, this holds for quasi all $s \in E$, i.e. off a left Haar null set.

Proof. Put $t_n := z_n^{-1}$, which is null. With $\sigma = \sigma(\mathbf{t})$ as in the Subcontinuity Theorem, since E is not left Haar null, there is g with $\sigma(gE) > 0$. For this g , put $\mu := {}_g\sigma$. Fix a compact $K_0 \subseteq E$ with $\mu(K_0) > 0$ and then, passing to a subsequence of \mathbf{t} as necessary (by Th. 1), we may assume that $\mu_{\mathbf{t}}^-(K_0) > 0$. Proceed to choose inductively a sequence $m(n) \in \mathbb{N}$, and decreasing compact sets $K_n \subseteq K_0 \subseteq E$ with $\mu(K_n) > 0$ such that

$$\mu(K_n \cap K_n t_{m(n)}) > 0.$$

To check the inductive step, suppose K_n already defined. As $\mu(K_n) > 0$, by the Subcontinuity Theorem, there is a subsequence $\mathbf{s} = \mathbf{s}(K_n)$ of \mathbf{t} with

$\mu_-^s(K_n) > 0$. By Lemma 1', there is $k(n) > n$ such that $\mu(K_n \cap K_n s_{k(n)}) > 0$. Putting $t_{m(n)} = s_{k(n)}$ and $K_{n+1} := K_n \cap K_n t_{m(n)} \subseteq K_n$ completes the inductive step.

By compactness, select s with

$$s \in \bigcap_{m \in \mathbb{N}} K_m \subseteq K_{n+1} = K_n \cap K_n t_{m(n)} \quad (n \in \mathbb{N});$$

choosing $k_n \in K_n \subseteq K$ with $s = k_n t_{m(n)}$ gives $t \in K_0 \subseteq E$, and

$$s z_{m(n)} = s t_{m(n)}^{-1} = k_n \in K_n \subseteq K_0 \subseteq E.$$

Finally take $\mathbb{M} := \{m(n) : n \in \mathbb{N}\}$.

As for the final assertion, we follow the idea of the Generic Completeness Principle [BinO1, Th. 3.4] (but with $\mathcal{U}(G)$ for $\mathcal{B}a$ there): define

$$F(H) := \bigcap_{n \in \mathbb{N}} \bigcup_{m > n} H \cap H t_m \quad (H \in \mathcal{U}(G));$$

then $F : \mathcal{U}(G) \rightarrow \mathcal{U}(G)$ and F is monotone ($F(S) \subseteq F(T)$ for $S \subseteq T$); moreover, $s \in F(H)$ iff $s \in H$ and $s z_m \in H$ for infinitely many m . We are to show that $E_0 := E \setminus F(E)$ is left Haar null. Suppose otherwise. Then renaming g and K_0 as necessary, w.l.o.g. both $\mu(E_0) > 0$ and $K_0 \subseteq E_0$ (and $\mu(K_0) > 0$). But then as above $\emptyset \neq F(K_0) \cap K_0 \subseteq F(E) \cap E_0$, a contradiction, since $F(E) \cap E_0 = \emptyset$. \square

Corollary 3. *For G Polish and amenable at 1_G and z_n null, there is $\mu \in \mathcal{P}(G)$ such that for $K \in \mathcal{K}(G) \cap \mathcal{M}_+(\mu)$*

$$K \cap K z_m^{-1} \in \mathcal{M}_+(\mu) \quad \text{for infinitely many } m \in \mathbb{N},$$

iff for μ -quasi all $s \in K$ there is an infinite $\mathbb{M} \subseteq \mathbb{N}$ with

$$\{s z_m : m \in \mathbb{M}\} \subseteq K.$$

Proof. We will refer to the function F of the preceding proof. First proceed as in the proof of Th. 3 above, taking $t_n := z_n^{-1}$ and $g = 1_G$ (so that $\mu = \sigma$). Fix K with $\mu(K) > 0$. For the forward direction, continue as in the proof of Th. 3 with $K_0 = K$ and observe that the proof above needs only that $s_{k(n)} \in K_n^{-1} K_n$ occurs infinitely often whenever $\mu(K_n) > 0$. This yields the desired conclusion that $\mu(K \setminus F(K)) = 0$. For the converse direction, suppose that $\mu(F(K)) > 0$. Since for each $n \in \mathbb{N}$

$$F(K) \subseteq \bigcup_{m > n} K \cap K t_m,$$

we have $\mu(K \cap Kt_m) > 0$ for some $m > n$; so

$$K \cap Kt_m \in \mathcal{M}_+(\mu) \quad \text{for infinitely many } m. \quad \square$$

Remark. With E as in the Shift-compactness Theorem, if $z_n \in B_{1/n} \setminus E^{-1}E$, then z_n is null; so, for some $s \in E$, $sz_m \in E$ for infinitely many m . Then, for any such m ,

$$z_m \in E^{-1}E,$$

contradicting the choice of z_m . So $1_G \in \text{int}(E^{-1}E)$, i.e. E has the Steinhaus-Weil property, as before.

The following sharpens a result due (for Lebesgue measure) to Mospan [Mos]; it is antithetical to Lemma 1 (and so to Theorem 3).

Proposition 6 (Mospan property). *For G a metric group, if $1_G \notin \text{int}(K^{-1}K)$, then $\mu_-(K) = 0$, i.e. μ is maximally discontinuous; equivalently, there is a ‘null sequence’ $t_n \rightarrow 1_G$ with $\lim_n \mu(Kt_n) = 0$.*

Conversely, if $\mu(K) > \mu_-(K) = 0$, then there is a null sequence $t_n \rightarrow 1_G$ with $\lim_n \mu(Kt_n) = 0$, and there is $C \subseteq K$ with $\mu(K \setminus C) = 0$ with $1_G \notin \text{int}(C^{-1}C)$.

Proof. The first assertion follows from Lemma 1. For the converse, as in [Mos]: suppose that $\mu(Kt_n) = 0$, for some sequence $t_n \rightarrow 1_G$. By passing to a subsequence, we may assume that $\mu(Kt_n) < 2^{-n-1}$. Put $D_m := K \setminus \bigcap_{n \geq m} Kt_n \subseteq K$; then $\mu(K \setminus D_m) \leq \sum_{n \geq m} \mu(Kt_n) < 2^{-m}$, so $\mu(D_m) > 0$ provided $2^{-m} < \mu(K)$. Now choose compact $C_m \subseteq D_m$, with $\mu(D_m \setminus C_m) < 2^{-m}$. So $\mu(K \setminus C_m) < 2^{1-m}$. Also $C_m \cap C_m t_n = \emptyset$, for each $n \geq m$, as $C_m \subseteq K$; but $t_n \rightarrow 1_G$, so the compact set $C_m^{-1}C_m$ contains no interior points. Hence, by Baire’s theorem, neither does $C^{-1}C$, since $C = \bigcup_m C_m$, which differs from K by a null set. \square

Proposition 7. *A (regular) Borel measure on a locally compact metric topological group G has the Steinhaus-Weil property iff*

- (i) *for each non-null compact set K , the map $m_K : t \rightarrow \mu(Kt)$ is subcontinuous at 1_G ;*
- (ii) *for each non-null compact set K there is no ‘null’ sequence $t_n \rightarrow 1_G$ with $\mu(Kt_n) \rightarrow 0$.*

Remark. This is immediate from Prop. 6 (cf. [Mos]).

We now prove a strengthening of the Subcontinuity Theorem obtained by assuming a ‘concentration property’. That this property holds in a abelian Polish group emerges from an inspection of Solecki’s proof of his theorem that an abelian Polish group is amenable at 1.

Definitions. Say that a null sequence \mathbf{t} is *regular* if \mathbf{t} is non-trivial, $\|t_k\|$ is non-increasing, and

$$\|t_k\| \leq r(k) := 1 / [2^k(k+1)] \quad (k \in \mathbb{N}).$$

For regular \mathbf{t} , put $\mu_k = \mu_k(\mathbf{t}) := 2^{k-1} \sum_{m \geq k} 2^{-m} (\delta_{t_m^{-1}} + \delta_{t_m}) = \frac{1}{2} \delta_{t_k^{-1}} + \frac{1}{4} \delta_{t_{k+1}^{-1}} + \dots$. Then $\mu_k(B_{r(k)}) = 1$.

If $\nu_k \ll \mu_k$, then

$$\nu_k := \sum_{m \geq k} a_{km} \delta_{t_m^{-1}},$$

for some non-negative sequence $\mathbf{a}_k := \{a_{kk}, a_{k,k+1}, a_{k,k+2}, \dots\}$ of unit ℓ_1 -norm. Say that $\{\mathbf{a}_k\}$ has the *concentration property* if for some index j and some $\alpha > 0$

$$a_{k,k+j} \geq \alpha > 0 \quad \text{for all large } k;$$

then say that the sequence $\{\nu_k\}$ has the *concentration property*. (This will fail if \mathbf{a}_k has $a_{k,k+k} = 1$, which concentrates measure in an unbounded fashion.)

Definition. Say that a group G is *strongly amenable at 1* if G is amenable at 1, and for each regular \mathbf{t} a Solecki measure $\sigma(\mathbf{t})$ has associated measures $\sigma_k(\mathbf{t}) \ll \mu_k(\mathbf{t})$ with the concentration property.

Theorem 4 (Strong amenability at 1, after [Sol2, Prop. 3.3(i)]). *Any abelian Polish group G is strongly amenable at 1.*

Proof. This follows Solecki’s construction of his reference measure in the case of $\mu_k(\mathbf{t})$ above. First define the normalized restriction

$$\sigma_k := \mu_k|_{B_{r(k)}} / \mu_k(B_{r(k)})$$

and then set

$$\sigma := \ast_{k=1}^{\infty} \rho_k \text{ for } \rho_k := \frac{1}{k+1} \sum_{i=0}^k \sigma_k^i.$$

(Convolution powers intended here.) Then Solecki shows that $\sigma_k * \sigma(K) \rightarrow \sigma(K)$ (for K compact). However, as \mathbf{t} is regular, $\mu_k \equiv \sigma_k$. But $a_{kk} = 1/2$ (all k), so the measures σ_k here have the concentration property. \square

Definitions (Sequence and measure symmetrization):

1. Merging \mathbf{t}^{-1} with \mathbf{t} by alternation of terms yields the regular sequence $\mathbf{s} = (s_1, s_2, \dots) := (t_1, t_1^{-1}, t_2, t_2^{-1}, \dots)$; we term this the **symmetrized sequence** of \mathbf{t} . (It is ‘symmetric’ in the sense only that $\|s_{2k-1}\| = \|s_{2k}\|$.)
2. For odd k , as $(\mu_k(\mathbf{t}) + \mu_k(\mathbf{t}^{-1}))/2$ is symmetric as a measure, taking $\sigma_{2k}(\mathbf{s}) = \sigma_{2k-1}(\mathbf{s}) := (\mu_k(\mathbf{t}) + \mu_k(\mathbf{t}^{-1}))/2$ in lieu of $\mu_k(\mathbf{t})$ above yields each ρ_k symmetric. So, in the abelian context of Theorem 4 above, the limiting convolution σ is a *symmetric Solecki measure* $\sigma(\mathbf{t})$.

Remark. Performing the symmetrization of the Definition above gives in the proof of Theorem 4 above that $a_{2k-1, 2k-1} = a_{2k-1, 2k} = a_{2k, 2k} = a_{2k, 2k+1} = 1/4$, which presents *simultaneous concentration* along \mathbf{t} and \mathbf{t}^{-1} .

We now re-run the proof of Theorem 1 with improved estimates to yield:

Theorem 1s (Strong Subcontinuity Theorem). *For G a Polish group that is strongly amenable at 1, if \mathbf{t} is regular and $\sigma = \sigma(\mathbf{t})$ is a Solecki measure – then for $K \in \mathcal{K}_+(\sigma)$*

$$\sigma(K) = \lim_n \sigma(Kt_n) = \sigma_-^{\mathbf{t}}(K).$$

Likewise, passing to the symmetrized sequence of \mathbf{t} as above and to a symmetric Solecki measure $\sigma(\mathbf{t})$ with the simultaneous concentration property (for \mathbf{t} and \mathbf{t}^{-1}) corresponding to an abelian context:

$$\sigma(K) = \lim_n \sigma(t_n K).$$

Proof. Fix \mathbf{t} and a corresponding Solecki reference measure $\nu(\mathbf{t})$ and its associated sequence ν_k , which as in Th. 4 has the concentration property. Write $\sigma_k := \sum_{m \geq k} a_{km} \delta_{t_m^{-1}}$; as σ_k has the concentration property, there are n_0, j and $\alpha > 0$ with

$$a_{kk+j} \geq \alpha > 0 \quad (k \geq n_0).$$

Now fix K compact with $\sigma(K) > 0$ and $\varepsilon > 0$. Put

$$\delta := \varepsilon / \left(\frac{2}{\alpha} - 1 \right) > 0,$$

as $\alpha \leq 1$. Then, by upper semicontinuity and by amenability at 1, there is $n_1 = n_1(\varepsilon, K) > n_0$ with

$$\sigma(Kt_k) \leq \sigma(K) + \delta \text{ and } \sigma_k * \sigma(K) \geq \sigma(K) - \delta \quad (k \geq n_1).$$

So (by upper semicontinuity) for $k \geq n_1$

$$\sum_{m \geq k, m \neq k+j} a_{km} \sigma(Kt_m) \leq \sum_{m \geq k, m \neq k+j} a_{km} (\sigma(K) + \delta) = (\sigma(K) + \delta)(1 - a_{kk+j}).$$

Also (by amenability at 1) for $k \geq n_1$

$$\begin{aligned} a_{kk+j} \sigma(Kt_{k+j}) &\geq \sigma(K) - \delta - \sum_{m \geq k, m \neq k+j} a_{km} \sigma(Kt_m) \\ &\geq \sigma(K) + \delta - 2\delta - (\sigma(K) + \delta)(1 - a_{kk+j}) \\ &= a_{kk+j} \sigma(K) - \delta(2 - a_{kk+j}). \end{aligned}$$

So for $m = k + j > n_1 + j$

$$\sigma(Kt_m) = \sigma(Kt_{k+j}) \geq \sigma(K) - \delta \left(\frac{2}{a_{kk+j}} - 1 \right) \geq \sigma(K) - \delta \left(\frac{2}{\alpha} - 1 \right) = \sigma(K) - \varepsilon.$$

As for the final assertion concerning symmetrization, note that $\sigma(t_n K) = \sigma(K^{-1} t_n^{-1}) \rightarrow \sigma(K^{-1}) = \sigma(K)$, by symmetry of σ . \square

We note an immediate corollary, needed in §4.

Corollary 4. *For G, \mathfrak{t} and ν as in Th. 1_S above, and $K, H \in \mathcal{K}(G)$, $\delta > 0$: if $0 < \Delta < \sigma(K)$ and $0 < D < \sigma(H)$, then there is n with*

$$B_\delta^{K\Delta} \cap B_\delta^{HD} \supseteq \{t_m : m \geq n\}.$$

Proof. Take $\varepsilon := \min\{\sigma(K) - \Delta, \sigma(H) - D\} > 0$. As $K, H \in \mathcal{K}_+(\sigma)$, there is n such that $\|t_m\| < \delta$ for $m \geq n$ and

$$\sigma(Kt_m) \geq \sigma(K) - \varepsilon \geq \Delta, \quad \sigma(Ht_m) \geq \sigma(H) - \varepsilon \geq D \quad (m \geq n). \quad \square$$

To accommodate varying sided-ness conventions, we close with the left-handed version of Theorem 1, in which $\tilde{\nu}_-^s(K)$ (defined at the start of the section) replaces $\nu_-^s(K)$.

Theorem 1_L (**Left Subcontinuity Theorem**, after Solecki [Sol2, Th. 1(ii)]). *For G amenable at 1_G , and \mathbf{t} a null sequence, there is $\sigma = \sigma_L(\mathbf{t}) \in \mathcal{P}(G)$ such that for each $K \in \mathcal{K}(G)$ with $\sigma(K) > 0$ there is a subsequence $\mathbf{s} = \mathbf{s}(K) := \{t_{m(n)}\}$ with*

$$\lim_n \sigma(t_{m(n)}K) = \sigma(K) \quad (n \in \mathbb{N}), \text{ so } \quad \tilde{\sigma}_-^s(K) > 0.$$

Proof. From Theorem 1 with $\sigma = \sigma(\mathbf{t})$, recalling that $\tilde{\sigma}(E) := \sigma(E^{-1})$ so that $\sigma(E) = \tilde{\sigma}(E^{-1})$, there is $\{t_{m(n)}\}$ with

$$\tilde{\sigma}(t_{m(n)}^{-1}K^{-1}) > (1 - 2^{-n})\tilde{\sigma}(K^{-1}) \quad (n \in \mathbb{N}), \text{ so } \quad \lim_n \tilde{\sigma}(t_{m(n)}^{-1}K^{-1}) > 0.$$

Since K^{-1} is compact and $t_{m(n)}^{-1}$ is null, replace K^{-1} by K , $t_{m(n)}^{-1}$ by $t_{m(n)}$; then with μ for $\tilde{\sigma}$

$$\mu(t_{m(n)}K) > (1 - 2^{-n})\mu(K) \quad (n \in \mathbb{N}), \text{ so } \quad \lim_n \mu(t_{m(n)}K) > 0. \quad \square$$

Part II: Around the Simmons-Mospan Theorem

We saw in Part I that measure may selectively exhibit continuity under translation in the presence of amenability at 1. Here we consider obstacles/obstructions to continuity.

3 A Lebesgue Decomposition

We begin with definitions isolating left-handed components in Christensen's notion of Haar null sets [Chr1], and Solecki's left Haar null sets [Sol2]; right-handed versions have analogous properties. As far as we are aware the component notions in parts (ii)-(iv) below have not been previously studied. Below G is a Polish group.

Definition. (i) **Left μ -null:** For $\mu \in \mathcal{M}(G)$, say that N is *left μ -null* ($N \in \mathcal{M}_0^L(\mu)$) if it is contained in a universally measurable set B such that

$$\mu(gB) = 0 \quad (g \in G).$$

Thus a set S is *left Haar null* ([Sol3] after [Chr1]) if it is contained in a universally measurable set B that is left μ -null for some $\mu \in \mathcal{M}(G)$.

(ii) **Left μ -inversion:** For $\mu \in \mathcal{M}(G)$, say that $N \in \mathcal{M}_0^L(\mu)$ is *left invertibly μ -null* ($N \in \mathcal{M}_0^{L\text{-inv}}(\mu)$) if

$$N^{-1} \in \mathcal{M}_0^L(\mu),$$

so that N^{-1} is contained in a universally measurable set B^{-1} such that

$$\mu(gB^{-1}) = 0 \quad (g \in G).$$

(iii) **Left μ -absolute continuity:** For $\mu, \nu \in \mathcal{M}(G)$, ν is *left absolutely continuous* w.r.t. μ ($\nu <^L \mu$) if $\nu(N) = 0$ for each $N \in \mathcal{M}_0^L(\mu)$, and likewise for the invertibility version: $\nu <^{L\text{-inv}} \mu$.

(iv) **Left μ -singularity:** For $\mu, \nu \in \mathcal{M}(G)$, ν is *left singular* w.r.t. μ (on B) ($\nu \perp^L \mu$ (on B)) if B is a support of ν and B is in $\mathcal{M}_0^L(\mu)$, and likewise $\nu \perp^{L\text{-inv}} \mu$.

Remark. For μ symmetric, since

$$g^{-1}\mu(B) = \mu_g(B^{-1})$$

if B is left μ -null we may conclude only that B^{-1} is *right μ -null*. The ‘inversion property’, property (ii) above, is thus quite strong (though obvious in the abelian case).

Notice that each of $\mathcal{M}_0^L(\mu)$ and $\mathcal{M}_0^{L\text{-inv}}(\mu)$ forms a σ -algebra (since $g \cup_n B = \cup_n gB$ and $g(\cup_n B)^{-1} = \cup_n gB^{-1}$). This implies the following left versions of the Lebesgue Decomposition Theorem (we need the second one below). The ‘pedestrian’ proof below demonstrates that the Principle of Dependent Choice (DC) suffices, a further example that ‘positive’ results in measure theory follow from DC (as Solovay points out in [Solo, p. 31]).

Theorem LD. *For G a Polish group, $\mu, \nu \in \mathcal{M}(G)$, there are $\nu_a, \nu_s \in \mathcal{M}(G)$ with*

$$\nu = \nu_a + \nu_s \text{ with } \nu_a <^L \mu \text{ and } \nu_s \perp^L \mu,$$

and likewise, there are $\nu'_a, \nu'_s \in \mathcal{M}(G)$ with

$$\nu = \nu'_a + \nu'_s \text{ with } \nu_a <^{\text{L-inv}} \mu \text{ and } \nu_s \perp^{\text{L-inv}} \mu.$$

Proof. As the proof depends on σ -additivity, it will suffice to check the ‘L’ case. Write $G = \bigcup_n G_n$ with the G_n disjoint, universally measurable, and with each $\nu(G_n)$ finite (say, with all but one term σ -compact, and their complement ν -null). Put $s_n = \sup\{\nu(E) : E \subseteq G_n, E \in \mathcal{M}_0^{\text{L}}(\mu)\}$. In $\mathcal{M}_0^{\text{L}}(\mu)$, for each n with $s_n > 0$, choose $E_{n,m} \subseteq G_n$ with $\nu(E_{n,m}) > s_n - 1/m$, and put $B_n := \bigcup_m E_{n,m} \subseteq G_n$. Then the sets B_n are disjoint and lie in $\mathcal{M}_0^{\text{L}}(\mu)$, as does also $B := \bigcup_n B_n$; moreover $\nu(G_n \setminus B_n) = 0$ for each n . Put $A := G \setminus B$. Then $\nu(M) = 0$ for $M \in \mathcal{M}_0^{\text{L}}(\mu)$ with $M \subseteq A$, since $A = \bigcup_n (G_n \setminus B_n)$. So $\nu_a := \nu|_A <^{\text{L}} \mu$, and $\nu_s := \nu|_{B \perp^{\text{L}} \mu}$, since $B \in \mathcal{M}_0^{\text{L}}(\mu)$. \square

Remark. The above rests on DC; a simpler argument rests on maximality: choose a maximal disjoint family \mathcal{B} of universally measurable sets $M \in \mathcal{M}_0^{\text{L}}(\mu)$ with finite positive $\nu(M)$; then, their union $B \in \mathcal{M}_0^{\text{L}}(\mu)$ (as \mathcal{B} would be countable, by the σ -finiteness of ν).

4 Discontinuity: the Simmons-Mospan Theorem

It is convenient to begin by repeating the gist of the Simmons-Mospan argument here, as it is short, despite its ‘near perfect disguise’, to paraphrase a phrase from Loomis [Loo, p. 85]. The result follows from their use of Fubini’s Theorem and the Lebesgue decomposition theorem of §3, but here we stress the dependence on Theorem FN and on left μ -inversion. We revert to the Weil left-sided convention and associated KK^{-1} usage.

Proposition 8 (Almost nullity). *For G a Polish group, $\mu \in \mathcal{M}(G)$, open V and compact K with $K \in \mathcal{M}_0^{\text{L-inv}}(\mu)$, i.e. $K, K^{-1} \in \mathcal{M}_0^{\text{L}}(\mu)$:
– for any $\nu \in \mathcal{M}(G)$, $\nu(tK) = 0$ for μ -almost all $t \in V$, and likewise $\nu(Kt) = 0$.*

Proof. For ν invertibly μ -absolutely continuous, the conclusion is immediate; for general ν this will follow from Theorem LD once we have proved the corresponding singular version of the assertion: that is the nub of the proof.

So suppose that $\nu \perp^{\text{L-inv}} \mu$ on K . Writing uw for t below, and noting that $\mu(uK^{-1}) = 0$, as $K^{-1} \in \mathcal{M}_0^{\text{L}}(\mu)$, the horizontal sections of the set H below are all μ -null:

$$\begin{aligned} H &:= \bigcup_{t \in V} (\{t\} \times tK) = \{(t, u) : u \in G, uw = t \in V, u \in uwK\} \\ &= \{(uw, u) : u \in G, uw \in V, w \in K^{-1}\} = \bigcup_{u \in G} (V \cap (uK^{-1})) \times \{u\}. \end{aligned}$$

By Theorem FN (§1), μ -almost all vertical t -sections for $t \in V$ are ν -null. As the assumptions on K are symmetric the right-sided version follows. \square

The result above is reminiscent of [Amb, Lemma 1.1]. Before stating the Simmons-Mospan specialization to the Haar context and also to motivate one of the conditions in its subsequent generalizations, we cite (and give a direct proof) of the following known result (equivalence of Haar measure η and its inverse $\tilde{\eta}$), encapsulated in the formula

$$\eta(K^{-1}) = \int_K d\eta(t)/\Delta(t),$$

exhibiting the direct connection between η and $\tilde{\eta}$ via the modular function [HewR, 15.14], or [Hal, §60.5f].

Lemma H (cf. [Hal, §50(ff); §59 Th. D]). *In a locally compact group G , for K compact, if $\eta(K) = 0$, then $\eta(K^{-1}) = 0$, and, by regularity, so also for K measurable.*

Proof. Fix a compact K . As K is compact, Δ (the modular function of G) is bounded away from 0 on K , say by $M > 0$; furthermore, K is separable, so pick $\{d_n : n \in \mathbb{N}\}$ dense in K . Then for any $\varepsilon > 0$ there are two (finite) sequences $m(1), \dots, m(n) \in \mathbb{N}$ and $\delta(1), \dots, \delta(n) > 0$ such that $\{B_{\delta(i)}d_{m(i)} : i \leq n\}$ covers K and

$$M \sum_{i \leq n} \eta(B_{\delta(i)}) \leq \sum_{i \leq n} \eta(B_{\delta(i)}) \Delta(d_{m(i)}) = \sum_{i \leq n} \eta(B_{\delta(n)}d_{m(n)}) < \varepsilon.$$

Then

$$\sum_{i \leq n} \eta(d_{m(i)}^{-1}B_{\delta(i)}) = \sum_{i \leq n} \eta(B_{\delta(i)}) \leq \varepsilon/M.$$

But $\{d_{m(i)}^{-1}B_{\delta(i)} : i \leq n\}$ covers K^{-1} by the symmetry of the balls B_δ (by the symmetry of the norm); so, as $\varepsilon > 0$ is arbitrary, $\eta(K^{-1}) = 0$.

As for the final assertion, if $\eta(E^{-1}) > 0$ for some measurable E , then $\eta(K^{-1}) > 0$ for some compact $K^{-1} \subseteq E^{-1}$, by regularity; then $\eta(K) > 0$, and so $\eta(E) > 0$. \square

Proposition 8 and Lemma H immediately give:

Proposition SM. *For G locally compact with left Haar measure η and ν a Borel measure on G , if the set S is η -null, then for η -almost all t*

$$\nu(tS) = 0.$$

In particular, this is so for S the support of a measure ν singular with respect to η .

For a locally compact group this in turn allows us to prove the separable case of the Simmons-Mospan Theorem, as stated in the Introduction. We then pursue a non-locally compact variant.

Proof of Theorem SM. If μ is absolutely continuous w.r.t. Haar measure η , then μ , being invariant, is subcontinuous, and Lemma 1 (in §2) gives the Steinhaus-Weil property. Otherwise, decomposing μ into its singular and absolutely continuous parts w.r.t. η , choose K a compact subset of the support of the singular part of μ ; then $\mu(K) > \mu_-(K) = 0$, by Prop. SM above, and so Prop. 6 (Converse part – see §2) applies. \square

Proposition 9 (after Simmons, cf. [Sim, Lemma]). *For G a Polish group, $\mu, \nu \in \mathcal{M}(G)$ and $\nu \perp^{\text{L-inv}} \mu$ concentrated on a compact invertibly μ -null set K , there is $B \subseteq K$ such that $K \setminus B$ is ν -null and both BB^{-1} and $B^{-1}B$ have empty interior.*

Proof. As we are concerned only with the subspace $KK^{-1} \cup K^{-1}K$, w.l.o.g. the group G is separable. By Prop. 8, $Z := \{x : \nu(xK) = 0\}$ is dense and so also

$$Z_1 := \{x : \nu(K \cap xK) = 0\},$$

since $\nu(K \cap xK) \leq \nu(xK) = 0$, so that $Z \subseteq Z_1$. Take a denumerable dense set $D \subseteq Z_1$ and put

$$S := \bigcup_{d \in D} K \cap dK.$$

Then $\nu(S) = 0$. Take $B = K \setminus S$. If $\emptyset \neq V \subseteq BB^{-1}$ and $d \in D \cap V$, then for some $b_1, b_2 \in B \subseteq K$

$$d = b_1 b_2^{-1} : \quad b_1 = db_2 \in K \cap dK \subseteq S,$$

a contradiction, since $B \cap S = \emptyset$. So $(K \setminus S)(K \setminus S)^{-1}$ has empty interior. A similar argument based on

$$T := \bigcup_{d \in D} Kd \cap K$$

ensures that also $(K \setminus S \setminus T)^{-1}(K \setminus S \setminus T)$ has empty interior. \square

In order to generalize the Simmons Theorem from its locally compact context we will need to cite the following result. Here $\mathbb{Q}_+ := \mathbb{Q} \cap (0, \infty)$ denotes the positive rationals, and $B_\delta^{K, \Delta}(\sigma) := \{z \in B_\delta : \sigma(Kz) > \Delta\}$ as in §2.

Theorem 5 (Disaggregation Theorem). *Let G be a Polish group that is strongly amenable at 1, and let \mathbf{t} be a regular null sequence. For $\sigma = \sigma(\mathbf{t})$ there are a countable family \mathcal{H} with $\mathcal{H} \subseteq \mathcal{K}_+(\sigma)$, a countable set $D = D(\mathcal{H}) \subseteq G$ dense in G , and a dense subgroup $G(\sigma)$ on which the sets below are the sub-basic sets of a metrizable topology:*

$$B_\delta^{K, \Delta}(\sigma) \quad (K \in \mathcal{H}, \delta, \Delta \in \mathbb{Q}_+, \Delta < \sigma(K)).$$

In particular, the space $G(\sigma)$ is continuously and compactly embedded in G . Moreover, each such open set contains a cofinal subsequence of \mathbf{t} .

For a proof we refer the reader to the Appendix of this paper; the result relies on Corollary 4 above. We are now ready for the promised generalization. This requires equivalence of the Solecki measure and its inverse – valid at least in Polish abelian groups (see Th. 4 of §2 on strong amenability at 1).

Theorem 6 (Generalized Simmons Theorem, cf. [Sim, Th. 2]). *Let G be a Polish group that is strongly amenable at 1 (e.g. if G is abelian), let $\sigma = \sigma(\mathbf{t})$ be a Solecki measure corresponding to a regular null sequence \mathbf{t} , which we assume is equivalent to its inverse $\tilde{\sigma}$ (e.g. if G is abelian), and let $G(\sigma)$ be the dense subgroup of the preceding theorem. Then: $\nu \in \mathcal{M}(G)$ is left invertibly-singular w.r.t. σ iff ν has a support that is a σ -compact union of compact sets K_n with each of the compact sets $K_n K_n^{-1}$ and*

$K_n^{-1}K_n$ nowhere dense (equivalently: having empty interior) in the topology of the subgroup $G(\sigma)$, as above.

Proof. Suppose that $\mu \in \mathcal{M}(G)$ is subcontinuous. If $\nu \in \mathcal{M}(G)$, by Theorem LD write

$$\nu = \nu_a + \nu_s \text{ with } \nu_a <^{\text{L-inv}} \mu \text{ and } \nu_s \perp^{\text{L-inv}} \mu.$$

If ν is concentrated on a σ -compact set B with $B^{-1}B$ having empty interior, then $\nu_a = 0$, and so ν is left invertibly-singular w.r.t. μ . Indeed, as ν is concentrated on B , so is ν_a . We claim that $\nu_a(B) = 0$. Otherwise, $\nu_a(K_n) > 0$ for some compact $K_n \subseteq B$. So $K = K_n \notin \mathcal{M}_0^{\text{L-inv}}(\sigma)$, as $\nu_a <^{\text{L-inv}} \sigma$. The argument now splits into two cases, according as $K \notin \mathcal{M}_0^{\text{L}}(\sigma)$ or $K^{-1} \notin \mathcal{M}_0^{\text{L}}(\sigma)$

First, suppose that $\sigma(gK) > 0$ for some $g \in G$; then, by Lemma 1, there are $\delta > 0$ and $0 < \Delta < \sigma_-^{\text{t}}(gK)$ with

$$B_\delta^{gK\Delta}(\sigma) \subseteq (gK)^{-1}gK = K^{-1}K \subseteq B^{-1}B,$$

contradicting the above property of B .

Next, suppose that $\sigma(Kg) = \sigma^{-1}(g^{-1}K^{-1}) > 0$ for some $g \in G$; so $\sigma(g^{-1}K^{-1}) > 0$. Then, again by Lemma 1, there are $\delta > 0$ and $0 < \Delta < \sigma_-^{\text{t}}(g^{-1}K^{-1})$ with

$$B_\delta^{g^{-1}K^{-1}\Delta}(\sigma) \subseteq (g^{-1}K^{-1})^{-1}g^{-1}K^{-1} = KK^{-1} \subseteq BB^{-1},$$

again contradicting the above property of B . So $\nu = \nu_a$ is invertibly singular w.r.t. μ .

The rest of the proof is as in Simmons [Sim, Th. 2], using Prop. 6: the Baire-category argument still holds, since compactness implies closure under the $G(\sigma(\mathbf{t}))$ -topology, the latter being a finer topology; avoidance of interior points requires second countability, assured by Th. 5. \square

Corollary 5 (Simmons Theorem: [Sim, Th. 2]). *For G separable and locally compact and η left Haar measure:*

$\nu \in \mathcal{M}(G)$ is singular w.r.t. η iff ν has a support that is a σ -compact union of compact sets K_n with each of the compact sets $K_n K_n^{-1}$ nowhere dense (equivalently: having empty interior).

Remark. Simmons establishes this theorem without separability by using the Kakutani-Kodaira ‘separable quotient’ approximation theorem for metric

groups [MonZ, I.2.6], cf. [HewR, Th. 8.7]. (Starting with a fixed compact support permits a reduction to the case of a compactly generated group, which is followed by working in a *separable* quotient group.)

Part III: Weil topologies

5 Weil-like topologies: preliminaries

We turn now to relatives of the *Weil topology*. For background, we refer to Weil's book [Wei, Ch. VII] and Halmos's book [Hal, Ch. XII]. (See also C.1 in the Appendix.) Weil regarded his result as a *Converse Haar Theorem*, in retrieving the topological-group structure from the measure-algebra structure [Fre] as encoded by the Haar-measurable subsets – cf. [Kod]. (Here one may work either, following Weil, to within a dense embedding in a locally compact group, as in the Remark to Theorem 7M below, or, following Mackey, uniquely up to homeomorphism, granted the further assumption of an analytic Borel structure [Mac, Th. 7.1].) The alternative view below throws light on this result in that the measure structure is already encoded by the density topology \mathcal{D} via the Haar density theorem, for which see [Mue], [Hal, §61(5), p. 268], cf. [BinO2, §7; Th. 6.10]; this view is partially implicit in [Amb], where refinement of one invariant measure μ_1 by another μ_2 holds when sets in $\mathcal{M}_+(\mu_2)$ contain sets in $\mathcal{M}_+(\mu_1)$ (as in the refinement of one topology by another). This falls within the broader aim of retrieving a *topological* group structure from a given (one-sidedly) invariant topology τ on a group G , when τ arises from refinement of a topological group structure (i.e. starting from a *semitopological* group structure (G, τ)). Also relevant here are *Converse Steinhaus-Weil* results, as in Prop. 7 of §3 above. (See also C.2 in the Appendix.) For background on group-norms see the textbook treatment in [ArhT, §3.3] (who trace this notion back to Markov) or [BinO2], but note their use of ‘pre-norm’ for what we call (following Pettis [Pet]) a *pseudo-norm*; for quasi-interiors and regular open sets see C.3. Thus a norm $\|\cdot\| : G \rightarrow [0, \infty)$ satisfies all the three conditions 1-3 below and generates a right-invariant metric $d(x, y) = \|xy^{-1}\|$ and so a topology $\mathcal{T} = \mathcal{T}_d$, just as a right-invariant metric d derives from a separable topology \mathcal{T}_G and generates, via the Birkhoff-Kakutani Theorem ([HewR, Th. 8.3], [Gao, Th. 2.1.1]),

the norm $\|x\| = d(x, 1_G)$. A pseudo-norm differs in possibly lacking 1.i. (so generates a *pseudo-metric*).

- 1.i (positivity): $\|g\| > 0$ for $g \neq 1_G$, and 1.ii: $\|1_G\| = 0$;
- 2 (subadditivity): $\|gh\| \leq \|g\| + \|h\|$,
- 3 (symmetry): $\|g^{-1}\| = \|g\|$.

Recall that a set function λ defined on $\mathcal{U}(G)$ is a *submeasure* if it is monotone and subadditive with $\lambda(\emptyset) = 0$ (Introduction, [Fre, Ch. 39, §392], [Tal]); by analogy with the term *finitely additive measure* (for background see [Bin], [Wag, Ch. 10]; cf. [Pat]), this is a *finitely subadditive outer measure*, similarly as in Maharam [Mah], albeit in the context of Boolean algebras, but without her positivity condition. Recall from Halmos [Hal, Ch. II §10] that a submeasure is an *outer measure* if in addition it is *countably subadditive*. The set function λ is *left invariant* if $\lambda(gE) = \lambda(E)$ for all $g \in G$ and $E \in \mathcal{U}(G)$.

Propositions 10 and 11 below are motivated by [Hal, Ch. XII §62, cf. Ch. II §9 (2-4)], where G is a locally compact group with λ its left Haar measure, but here the context is broader, allowing in *amenable* groups G (cf. [Wag, Ch. 10], [Pat]). The two results enable the introduction in §5 of Weil-like topologies generated from families of left-invariant *pseudo-metrics* derived from invariant submeasures. The latter rely on the natural *measure-metric*, also known as the *Fréchet-Nikodym metric* ([Fre, §323Ad], [Hal, §40 Th. A], [Bog1, p. 53, 102-3, 418]); see [Drew1,2] (cf. [Web]) for the related literature of Fréchet-Nikodym topologies and their relation to the Vitali-Hahn-Saks Theorem. Maharam [Mah] studies sequential continuity of the order relation (of inclusion, here in the measure algebra), and requires positivity to obtain a measure *metric*; see Talagrand [Tal] (cf. [Fre, §394] and the literature cited there) for a discussion of pathological submeasures (the only measures they dominates under \ll being trivial), and [ChrH] for corresponding exotic abelian Polish groups.

In the setting of a locally compact group G , these pseudo-metrics are implicit in work of Struble: initially, in 1953 [Str1], he used a ('sampler') family of pre-compact open sets $\{E_t : t > 0\}$ to construct a mean on G , thereby referring to a one-parameter family of pseudo-metrics corresponding to the sets E_t ; some twenty years later in 1974 [Str2] (cf. [DieS, Ch. 8]) identifies a left-invariant (proper) metric on G by taking the supremum of pseudo-metrics, each generated from some open set in a countable open base at 1_G .

Proposition 10 (Weil pseudo-norm, cf. [Fre, §392H]). *For G a Polish group, $\lambda \in \mathcal{M}_{\text{sub}}(G)$, a left-invariant submeasure on $\mathcal{U}(G)$, and $E \in \mathcal{U}(G)$ with $\lambda(E) > 0$, put*

$$\|g\|_E^\lambda := \lambda(gE \triangle E) \quad (g \in G).$$

Then $\|\cdot\|_E$ defines a group pseudo-norm with associated right-invariant pseudo-metric

$$d_E^\lambda(g, h) = \|gh^{-1}\|_E^\lambda \quad (g, h \in G).$$

Likewise, for λ right-invariant, a pseudo-norm is defined by

$$\|g\|_E^\lambda := \lambda(E \triangle Eg) \quad (g \in G).$$

Proof. Since $\lambda(\emptyset) = 0$, $\|1_G\|_E^\lambda = 0$. By left invariance under a ,

$$\|a^{-1}\|_E^\lambda = \lambda(a^{-1}E \triangle E) = \lambda(a(a^{-1}E \triangle E)) = \lambda(E \triangle aE) = \|a\|_E^\lambda.$$

Also,

$$\|ab\|_E^\lambda \leq \|a\|_E^\lambda + \|b\|_E^\lambda$$

follows from monotonicity, subadditivity and $\lambda(abE \triangle aE) = \lambda(bE \triangle E)$:

$$\begin{aligned} \lambda(abE \setminus E \cup E \setminus abE) &\leq \lambda(abE \setminus aE) \cup (aE \setminus E) \cup (E \setminus aE) \cup (aE \setminus abE)) \\ &= \lambda(abE \setminus aE) \cup (aE \setminus abE) \cup (aE \setminus E) \cup (E \setminus aE)) \\ &\leq \lambda(abE \triangle aE) + \lambda(E \triangle aE) = \lambda(bE \triangle E) + \lambda(E \triangle aE). \end{aligned} \quad \square$$

Recall now (from the Introduction) that a subset of a Polish group G is *left Haar null* if it is contained in a universally measurable set B such that for some $\mu \in \mathcal{P}(G)$

$$\mu(gB) = 0 \quad (g \in G).$$

It is *Haar null* [Sol1] (cf. [HofT, p. 374]) if it is contained in a universally measurable set B such that for some $\mu \in \mathcal{P}(G)$

$$\mu(gBh) = 0 \quad (g, h \in G).$$

This motivates the following application of Proposition 10 beyond Haar measure. Extending the notation of §3, below $\mathcal{M}_0^L(G)$ (resp. $\mathcal{M}_0(G)$) denotes the family of left-Haar-null (resp. Haar-null) sets of G , and we write

$$\mathcal{U}_+^L(G) := \mathcal{U}(G) \setminus \mathcal{M}_0^L(G), \quad \mathcal{U}_+(G) := \mathcal{U}(G) \setminus \mathcal{M}_0(G).$$

Proposition 11. *In a Polish group G , for $\mu \in \mathcal{P}(G)$ put*

$$\begin{aligned}\mu_L^*(E) &: = \sup\{\mu(gE) : g \in G\} & (E \in \mathcal{U}(G)), \\ \hat{\mu}(E) &: = \sup\{\mu(gEh) : g, h \in G\} & (E \in \mathcal{U}(G)).\end{aligned}$$

Then μ_L^ (resp. $\hat{\mu}$) is a left invariant (resp. bi-invariant) submeasure, which is positive for $E \in \mathcal{U}_+^L(G)$ (resp. for $E \in \mathcal{U}_+(G)$), i.e. for universally measurable, non-left Haar null (resp. non-Haar null) sets.*

Proof. We consider only $\hat{\mu}$, as the case μ_L^* is similar and simpler (through the omission of h and b below). The set function $\hat{\mu}$ is well defined, with

$$\mu(E) \leq \hat{\mu}(E) \leq 1 \quad (E \in \mathcal{U}(G)),$$

since μ is a probability measure; it is bi-invariant, since

$$\hat{\mu}(aEb) := \sup\{\mu(gaEbh) : g, h \in G\} = \sup\{\mu(gEh) : g, h \in G\},$$

and G is a group. Furthermore, for $B \in \mathcal{U}(G)$

$$\mu(gBh) \leq \hat{\mu}(B) \leq 1, \quad (g, h \in G).$$

So, for $\mu \in \mathcal{P}(G)$

$$0 < \hat{\mu}(B) \leq 1 \quad (B \in \mathcal{U}_+(G)),$$

since there are $g, h \in G$ with $\mu(gBh) > 0$. Countable subadditivity follows (on taking suprema of the leftmost term over g, h) from

$$\mu(g(\bigcup_n A_n)h) \leq \sum_n \mu(gA_nh) \leq \sum_n \hat{\mu}(gA_nh) = \sum_n \hat{\mu}(A_n),$$

for any sequence of sets $A_n \in \mathcal{U}(G)$. \square

For $\mu \in \mathcal{P}(G)$, $E \in \mathcal{U}(G)$, put

$$B_\varepsilon^E(\mu) := \{x \in G : \|x\|_E^\mu < \varepsilon\}.$$

Our next step is to inscribe these balls into EE^{-1} for all small enough $\varepsilon > 0$.

Lemma 2 (Self-intersection Lemma). *In a Polish group G for $E \in \mathcal{U}_+(G)$, and respectively for $E \in \mathcal{U}_+^L(G)$, and $\mu \in \mathcal{P}(G)$,*

$$\left. \begin{aligned} 1_G &\in B_\varepsilon^E(\hat{\mu}) \subseteq EE^{-1} & (0 < \varepsilon < \hat{\mu}(E)), \\ 1_G &\in B_\varepsilon^E(\mu_L^*) \subseteq EE^{-1} & (0 < \varepsilon < \mu_L^*(E)). \end{aligned} \right\}$$

Equivalently, for $0 < \varepsilon < \hat{\mu}(E)$, and respectively for $0 < \varepsilon < \mu_L^*(E)$,

$$E \cap xE \neq \emptyset \quad (x \in B_\varepsilon^E(\hat{\mu})); \quad E \cap xE \neq \emptyset \quad (x \in B_\varepsilon^E(\tilde{\mu})).$$

Proof. We check only the $\hat{\mu}$ case; the other is similar and simpler (through the omission of h below). For $E \in \mathcal{U}_+(G)$, since $\hat{\mu}(E) > 0$, we may pick $g, h \in G$ such that $\varepsilon_E := \mu(gEh) > 0$. Consider x and $\varepsilon > 0$ with $\|x\|_E^\mu < \varepsilon \leq \varepsilon_E$. If E and xE are disjoint, then

$$\begin{aligned} \varepsilon_E &= \mu(gEh) \leq \mu(g(E \cup xE)h) \leq \hat{\mu}(g(E \cup xE)h) = \hat{\mu}(E \cup xE) \\ &= \hat{\mu}(xE \triangle E) = \|x\|_E^\mu < \varepsilon \leq \varepsilon_E, \end{aligned}$$

a contradiction. So E and xE do meet. Now first pick $t \in xE \cap E$ and next $s \in E$ so that $t = xs$; then $x = ts^{-1} \in EE^{-1}$. The argument is valid for any ε_E with ε_E taking any value $\mu(gEh)$ in $(0, \hat{\mu}(E)]$, as in Prop. 1M. For the converse, if $B_\varepsilon^E(\hat{\mu}) \subseteq EE^{-1}$, proceed as in Prop. 1M. \square

We need a simple analogue of a result due to Weil ([Wei, Ch. VII, §31], cf. [Hal, Ch. XII §62]). Below τ_1 denotes the τ -open *neighbourhoods* of 1_G . For G locally compact and $\lambda = \eta$, Haar measure, the identity

$$2\eta(E) - 2\eta(E \cap xE) = \eta(E \triangle xE) = 1 - 2 \int 1_E(t) 1_{E^{-1}}(t^{-1}x) d\eta(t) \quad (\dagger)$$

connects the continuity of the (pseudo-) norm to \mathcal{T}_d -continuity of translation in the topological group structure (G, \mathcal{T}_d) of the locally compact group, and to continuity of the convolution function here (for E of finite η -measure) – see [HewR, Th. 20.16]; see also [HewR, Th. 20.17] for the well-known connection between the Steinhaus-Weil Theorem and convolution. Such a continuity condition result guarantees that $B_\varepsilon^E(\eta)$ contains points other than 1_G .

Lemma 3 (Fragmentation Lemma; cf. [Hal, Ch. XII §62 Th. A]). *For λ a left-invariant submeasure on a Polish group G equipped with a finer right-invariant topology τ with $\tau_1 \subseteq \mathcal{U}_+^L(G)$:*

if the map

$$x \mapsto \|x\|_E^\lambda$$

is continuous under τ at $x = 1_G$ for each $E \in \mathcal{U}_+^L(G)$

– then, for each $\emptyset \neq E, F \in \tau$ and $\varepsilon > 0$ with $\varepsilon < \lambda(E)$, there exists $H \in \tau_1$ with $HH^{-1} \subseteq FF^{-1}$ and

$$\|h'h^{-1}\|_E^\lambda < \varepsilon \quad (h, h' \in H) : \quad HH^{-1} \subseteq B_\varepsilon^E,$$

so that $\text{diam}_E^\lambda(H) \leq \varepsilon$.

Proof. Pick any $f \in F$, and $D \in \tau_1$ satisfying $\|x\|_E^\lambda < \varepsilon/2$ for all $x \in D$. As τ is right-invariant and $1_G \in D \cap Ff^{-1} \in \tau$, pick $H \in \tau_1$ with $H \subseteq D \cap Ff^{-1}$; then

$$HH^{-1} = Hf f^{-1} H^{-1} \subseteq FF^{-1}.$$

For $h, h' \in H$, as $h, h' \in D$

$$\|h'f(hf)^{-1}\|_E^\lambda = \|h'h^{-1}\|_E^\lambda \leq \|h'\|_E^\lambda + \|h^{-1}\|_E^\lambda = \|h'\|_E^\lambda + \|h\|_E^\lambda < \varepsilon. \quad \square$$

In the presence of a refinement topology τ on the group G , the lemma motivates further notation: $\mathcal{P}_{\text{cont}}(G, \tau)$, or just

$$\mathcal{P}_{\text{cont}}(\tau) := \{\mu \in \mathcal{P}(G, \mathcal{E}) : g \mapsto \|g\|_E^{\hat{\mu}} := \hat{\mu}(gE \triangle E) \text{ is } \tau\text{-continuous at } 1_G\}.$$

Attention here focuses of necessity on continuity. The characterization question as to which topologies τ yield a non-empty $\mathcal{P}(\tau)$ is in part answered by Theorem 7M below. Indeed, for Haar measure η in the locally compact case,

$$\mu \in \mathcal{P}_{\text{cont}}(\tau) \quad (\mu \ll \eta, \tau \supseteq \mathcal{E}),$$

by (\dagger) in the presence of $d\mu/d\eta$ as a kernel:

$$\|x\|_E^\mu = 1 - 2 \int 1_E(t) 1_{E^{-1}}(t^{-1}x) \frac{d\mu}{d\eta}(t) d\eta(t). \quad (\dagger\dagger)$$

However, $\mathcal{P}(G)$ will contain measures μ singular with respect to η : for such μ , by Th. SM there will be Borel subsets B of positive μ -measure such that BB^{-1} has void \mathcal{T}_d -interior.

6 Weil-like topologies: theorems

Prop. 11 now yields the following result, which embraces known Hashimoto topologies [BinO5] in both the Polish abelian setting, where the left Haar null sets form a σ -ideal (Christensen [Chr1]), and likewise in (the not necessarily abelian) Polish groups that are *amenable at 1* (Solecki [Sol1]); this includes, as additive groups, F - (hence Banach) spaces – cf. [BinO4,5], where use is made of Hashimoto topologies.

Theorem 7. *Let G be a Polish group and τ both a left- and a right-invariant refinement topology with $\tau_1 \subseteq \mathcal{U}_+(G)$. Then both the families $\{AA^{-1} : A \in \tau_1\}$ and $\{B_\varepsilon^E(\hat{\mu}) : \emptyset \neq E \in \tau, \mu \in \mathcal{P}(\tau) \text{ and } 0 < \varepsilon \leq \hat{\mu}(E)\}$ generate neighbourhoods of the identity under which G is a topological group. Moreover, the pseudo-norms*

$$\{\|\cdot\|_E^\mu : \emptyset \neq E \in \tau, \mu \in \mathcal{P}_{\text{cont}}(\tau)\}$$

are downward directed by refinement; indeed, for $\emptyset \neq E, F \in \tau_1$, $\lambda, \mu \in \mathcal{P}(\tau)$ and $\varepsilon < \min\{\hat{\lambda}(E), \hat{\mu}(F)\}$, there is $H \in \tau_1$ such that for $0 < \delta < \min\{\hat{\lambda}(H), \hat{\mu}(H)\}$

$$B_\delta^H(\lambda) \cap B_\delta^H(\mu) \subseteq B_\varepsilon^E(\lambda) \cap B_\varepsilon^F(\mu).$$

Proof. The proof is similar to but simpler than that of [Hal, Ch. XII §62 Th. A]. Note that $\tau \setminus \{\emptyset\} \subseteq \mathcal{M}_+$, and that, since τ refines the usual topology, for any $g \neq 1_G$ there is a usual, non-empty, open $E \in \tau \cap \mathcal{M}_+$ with $gE \cap E = \emptyset$, so $\|g\|_E = \eta(gE \triangle E) = 2\eta(E) > 0$.

Given two (non-left-Haar-null) sets $E, F \in \tau_1$ and $\varepsilon < \min\{\hat{\lambda}(E), \hat{\mu}(F)\}$, by the Fragmentation Lemma (Lemma 3 of §5) applied separately to $\hat{\lambda}$ and to $\hat{\mu}$, there are $A, B \in \tau_1$ with

$$AA^{-1} \subseteq B_\varepsilon^E(\hat{\lambda}), \quad BB^{-1} \subseteq B_\varepsilon^F(\hat{\mu}).$$

Take $H \in \tau_1$ with $H \subseteq A \cap B$; then

$$HH^{-1} \subseteq AA^{-1} \cap BB^{-1}.$$

Since $H \in \mathcal{U}_+(G)$ (as $\tau_1 \subseteq \mathcal{U}_+(G)$), take δ with $0 < \delta < \min\{\hat{\lambda}(H), \hat{\mu}(H)\}$; then by (*)

$$B_\delta^H(\hat{\lambda}) \cap B_\delta^H(\hat{\mu}) \subseteq HH^{-1} \subseteq AA^{-1} \cap BB^{-1} \subseteq B_\varepsilon^E(\hat{\lambda}) \cap B_\varepsilon^F(\hat{\mu}).$$

(So ‘mutual refinement’ holds between the sets of the form AA^{-1} and those of the form B_ε^E .) As $\|\cdot\|_E^\mu$ is a pre-norm,

$$B_{\varepsilon/2}^E(\hat{\mu})B_{\varepsilon/2}^E(\hat{\mu})^{-1} = B_{\varepsilon/2}^E(\hat{\mu})B_{\varepsilon/2}^E(\hat{\mu}) \subseteq B_\varepsilon^E(\hat{\mu}).$$

By the Fragmentation Lemma again, given any $x \in G$ and $\varepsilon > 0$, choose $H \in \tau_1$ with $HH^{-1} \subseteq B_\varepsilon^E(\hat{\mu})$. Then with $F := xH \in \tau$,

$$B_\varepsilon^F(\hat{\mu}) = \{z : \|z\|_F^\mu < \varepsilon\} \subseteq (xH)(xH)^{-1} = xHH^{-1}x^{-1} \subseteq xB_\varepsilon^E(\hat{\mu})x^{-1}.$$

Finally, for any x_0 with $\|x_0\|_E^{\hat{\mu}} < \varepsilon$, put $\delta := \varepsilon - \|x_0\|_E^{\hat{\mu}}$. Then for $\|y\|_E^{\hat{\mu}} < \delta$,

$$\|x_0 \cdot y\|_E^{\hat{\mu}} \leq \|x_0\|_E^{\hat{\mu}} + \|y\|_E^{\hat{\mu}} < \|x_0\|_E^{\hat{\mu}} + \varepsilon - \|x_0\|_E^{\hat{\mu}} < \varepsilon,$$

i.e.

$$x_0 B_\delta^E(\hat{\mu}) \subseteq B_\varepsilon^E(\hat{\mu}). \quad \square$$

Specializing to locally compact groups yields as a corollary, on writing $B_\varepsilon^E := B_\varepsilon^E(\eta)$:

Theorem 7M. *For G a locally compact group with left Haar measure η , if:*
(i) τ is both a left- and a right-invariant refinement topology with $\tau_1 \subseteq \mathcal{M}_+$,
(ii) for every non-empty $E \in \tau$, the pseudo-norm

$$g \mapsto \|g\|_E := \eta(gE\Delta E) \quad (g \in G)$$

is continuous under τ at $g = 1_G$

– then both the families $\{AA^{-1} : A \in \tau_1\}$ and $\{B_\varepsilon^E : \emptyset \neq E \in \tau \text{ and } 0 < \varepsilon \leq 2\eta(E)\}$ generate neighbourhoods of the identity under which G is a topological group. Moreover, the pseudo-norms

$$\{\|\cdot\|_E : \emptyset \neq E \in \tau\}$$

are downward directed by refinement; indeed, for $\emptyset \neq E, F \in \tau$ and $\varepsilon < 2\min\{\eta(E), \eta(F)\}$, there is $H \in \tau_1$ such that for $0 < \delta < \eta(H)$

$$B_\delta^H \subseteq B_\varepsilon^E \cap B_\varepsilon^F.$$

Proof. It is enough to replace $\mathcal{P}(G)$ by $\{\eta\}$ (so that λ and μ both refer to η), and to note that if xE and E are disjoint, then $\eta(xE\Delta E) = 2\eta(E)$, so that in Lemma 2 the bound $\eta^*(E)$ in the restriction governing inclusion may be replaced by $2\eta(E)$. \square

Remark. As in [Hal, Ch. XII §62 Th. F], but by the Fragmentation Lemma (and by the countable additivity of η), the Weil-like topology of Theorem 7M is locally bounded (norm-totally-bounded in some ball). So G with the Weil-like topology may be densely embedded in its completion \hat{G} , which is in turn locally compact, being locally complete and (totally) bounded. However, the corresponding argument in the case of the main theorem (Theorem 7) fails, since $\hat{\mu}$ there is not necessarily countably additive.

Finally, we give a category version of Theorem 7M, as an easy corollary; indeed, our main task is merely to define what is meant by ‘mutatis mutandis’ in the present context. Given the assumption $\tau_1 \subseteq \mathcal{B}_+$, we are entitled to refer to the usual quasi-interior of any $E \in \mathcal{B}_+$, denoted below by \tilde{E} , as in Cor. 2’; we also write \tilde{B}_ε^E for $B_\varepsilon^{\tilde{E}}(\eta)$.

Theorem 7B. *For G a locally compact group with left Haar measure η , if:*
(i) τ is both a left- and a right-invariant refinement topology with $\tau_1 \subseteq \mathcal{B}_+$ and with the left Nikodym property (preservation of category under left shifts),
(ii) for every non-empty $E \in \tau$ the pseudo-norm

$$g \mapsto \|g\|_{\tilde{E}} := \eta(g\tilde{E}\Delta\tilde{E}) \quad (g \in G)$$

is continuous under τ at $g = 1_G$
– then both the families $\{AA^{-1} : A \in \tau_1\}$ and $\{\tilde{B}_\varepsilon^E : \emptyset \neq E \in \tau \text{ and } 0 < \varepsilon \leq 2\eta(\tilde{E})\}$ generate neighbourhoods of the identity under which G is a topological group. Moreover, the pseudo-norms

$$\{\|\cdot\|_{\tilde{E}} : \emptyset \neq E \in \tau\}$$

are downward directed by refinement; indeed, for $\emptyset \neq E, F \in \tau$ and $\varepsilon < 2\min\{\eta(\tilde{E}), \eta(\tilde{F})\}$, there is $H \in \tau_1$ such that for $0 < \delta < 2\eta(\tilde{H})$

$$\tilde{B}_\delta^H \subseteq \tilde{B}_\varepsilon^E \cap \tilde{B}_\varepsilon^F.$$

Proof. In place of the inclusion of Lemma 2 we note a result stronger than for \tilde{E} (i.e. inclusion only in $\tilde{E}\tilde{E}^{-1}$): since meagreness is translation-invariant (the ‘Nikodym property’ of [BinO5]), $(xE)^\sim = x\tilde{E}$ for non-meagre Baire E , so $x\tilde{E} \cap \tilde{E} \neq \emptyset$ implies $xE \cap E \neq \emptyset$, and so again

$$B_\varepsilon^{\tilde{E}} \subseteq EE^{-1};$$

here again in Lemma 2 the bound $\eta^*(E)$ in the restriction governing inclusion may be replaced by $2\eta(E)$. The proof of Theorem 7 may now be followed verbatim, but for replacing $\mathcal{P}(G)$ by $\{\eta\}$, using the stronger inclusion just observed, and $B_\varepsilon(\eta)$ by \tilde{B}_ε . \square

Remark. The last result follows more directly from Th. 7M in a context where there exists on G a *Marczewski measure* (see [Wag, Ch. 11, cf. 9.8, 9.9]), i.e. a finitely additive invariant measure on \mathcal{B} vanishing on bounded

members of \mathcal{B}_0 ; this includes $\mathbb{R}, \mathbb{R}^2, \mathbb{S}^1$, albeit under AC [Wag, Cor. 11.3]; cf. [Myc].

With the groundwork of Part I on translation-continuity for *compacts* completed, we close by establishing the promised dichotomy associated with the map

$$x \mapsto \|x\|_E^\mu = \mu(xE \triangle E),$$

for measurable E : Theorem FN (§1) creates a duality between the vanishing of the F -based pseudo-norm and a *dichotomy* for x -translates of E^{-1} in relation to F according as $x \in E$ or $x \notin E$, which are thus unable in each case to distinguish between the points of F . Below we write \forall^μ for the generalized quantifier “for μ -a.a.” (cf. [Kec, 8.J]).

Theorem 8 (Almost Inclusion-Exclusion). *For G a Polish group $\mu \in \mathcal{P}(G)$ and non-null μ -measurable E, F , the vanishing μ -a.e. on F of the E -norm under μ :*

$$\|x\|_F^\mu = \mu(xE \triangle E) = 0 \quad (x \in F),$$

is equivalent to the following Almost Inclusion-Exclusion for translates of E^{-1} :

(i) Inclusion: F is μ -almost covered by μ -almost every translate xE^{-1} for $x \in E$:

$$\mu(F \setminus xE^{-1}) = 0 \quad (\forall^\mu x \in E),$$

(ii) Exclusion: F is μ -almost disjoint from μ -almost every translate xE^{-1} for $x \notin E$:

$$\mu(F \cap xE^{-1}) = 0 \quad (\forall^\mu x \notin E).$$

Proof. By Theorem FN (§1), applied to the set H of Prop. 3, i.e.

$$H := \bigcup_{x \in F} \{x\} \times (xE \triangle E),$$

H has vertical sections H_x almost all μ -null iff μ -almost all of its horizontal sections H^y are μ -null. But, since $y \in xE$ iff $x \in yE^{-1}$, $H^y = F \setminus yE^{-1}$ for $y \in E$ and $H^y := F \cap yE^{-1}$ for $y \in G \setminus E$. \square

Remark. If the inclusion side of the dichotomy of Th. 8 holds for all $x \in E$, then $F \subseteq EE^{-1}$. The converse direction may fail: consider $E = (1, 2)$ and $F = (-1, 1)$, so that $E - E = F$, but no translate of $-E$ may cover F .

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Appendix

Complements

C1. *Inclusion-Exclusion dichotomy*. Our Weil-like analysis in Part II focuses on inclusions amongst sets of the form EE^{-1} , the exception being the

Inclusion-Exclusion of a set F by an E - or non- E -translate of E^{-1} in Theorem 8 (a dichotomy as between E and its complement). This places most of our study on one side of a related inclusion-exclusion dichotomy, for subsets H, B of a group G one has either inclusion, or ‘near-disjointness’:

$$HH^{-1} \subseteq BB^{-1}, \quad \text{or} \quad HH^{-1} \cap BB^{-1} = \{1_G\}.$$

Inclusion may be equivalently re-phrased to the meeting of distinct pairs of H^{-1} -translates of B :

$$kB \cap k'B \neq \emptyset \quad (k, k' \in H^{-1}), \quad (\text{I})$$

whereas exclusion to their disjointness:

$$kB \cap k'B = \emptyset \quad (\text{distinct } k, k' \in H^{-1}). \quad (\text{E})$$

The duality of the relation of (E) to the results in Th. 8 is clarified by observing that $\mu(F \cap xE^{-1}) = 0$, for a.a. $x \in C$, is equivalent to $\mu(C \cap yE) = 0$, for a.a. $y \in F$; indeed

$$0 = \int \int 1_C(x) 1_F(y) 1_{xE^{-1}}(y) d(\mu \times \mu) = \int \int 1_F(y) 1_C(x) 1_{yE}(x) d(\mu \times \mu).$$

The condition (E) gives rise to \mathcal{I}_0 , the σ -ideal introduced in Balcerzak et al. [BalRS], generated by Borel sets B having perfectly many disjoint translates, as in (E) above with H^{-1} a perfect compact set (i.e. compact and dense-in-itself); continuum-many disjoint translates of a compactum also emerge in a theorem of Ulam concerning a non-locally compact Polish group: see [Oxt1, Th. 1]. Such ‘perfect’ exclusions offer a combinatorial tool, akin to *shift-compactness* (as in Th. 3, subsequence embedding under translation of a null sequence into a non-negligible set – cf. [BinO2,3] [MilO]), and play a key role in the context of groups with ‘ample generics’; see for instance the small-index property of [HodHLS].

Solecki [Sol3] proves a ‘Fubini for negligibles’-type theorem (cf. Theorem FN in §1 above): the non-negligible vertical sections (relative to a uniformly Steinhaus ideal) of a planar \mathcal{I}_0 -negligible set form a horizontal \mathcal{I}_0 -negligible set. The ideal \mathcal{I}_0 is of interest, as it violates the countable (anti)-chain condition, [BalRS].

C2. Steinhaus-Weil property of a Borel measure. In a locally compact group G , the family $\mathcal{M}_+(\mu)$ of finite non-null measurable sets of a Borel measure μ

on G fails to have the Steinhaus-Weil property iff there are a null sequence $z_n \rightarrow 1_G$ and a non-null compact set K with $\lim_n \mu(t_n K) = 0$. Equivalently, this is so iff the measure μ is not absolutely continuous with respect to Haar measure: these observations motivated Th. SM.

C3. *Regular open sets.* Recall that U is regular open if $U = \text{int}(\text{cl}U)$, and that $\text{int}(\text{cl}U)$ is itself regular open; for background see e.g. [GivH, Ch. 10]. For $\mathcal{D} = \mathcal{D}_{\mathcal{B}}$ the Baire-density topology of a normed topological group, let $\mathcal{D}_{\mathcal{B}}^{RO}$ denote the regular open sets. For $D \in \mathcal{D}_{\mathcal{B}}^{RO}$, put

$$N_D := \{t \in G : tD \cap D \neq \emptyset\} = DD^{-1}, \quad \mathcal{N}_1 := \{N_D : 1_G \in D \in \mathcal{D}_{RO}\};$$

then \mathcal{N}_1 is a base at 1_G (since $1_G \in C \in \mathcal{D}_{RO}$ and $1_G \in D \in \mathcal{D}_{RO}$ yield $1_G \in C \cap D \in \mathcal{D}_{RO}$) comprising \mathcal{T} -neighbourhoods that are $\mathcal{D}_{\mathcal{B}}$ -open (since $DD^{-1} = \bigcup \{Dd^{-1} : d \in D\}$). We raise the (metrizability) question, by analogy with the Weil topology of a measurable group (see §5 and C.1 above): with $\mathcal{D}_{\mathcal{B}}$ above replaced by a general density topology \mathcal{D} on a group G , when is the topology generated by \mathcal{N}_1 on G a norm topology? Some indications of an answer may be found in [ArhT, §3.3]. We note the following answer in the context of Theorem 7B; compare Struble's Theorem [Str2], or [DieS, Ch. 8]. If there exists a separating sequence D_n , i.e. such that for each $g \neq 1_G$ there is n with $\|g\|_{\tilde{D}_n} = 1$, then

$$\|g\| := \sum_n 2^{-n} \|g\|_{\tilde{D}_n}$$

is a norm, since it is separating and, by the Nikodym property, $(D \cap g^{-1}D) = g^{-1}(gD \cap D) \in \mathcal{B}_0$.

C4. *The Effros Theorem* asserts that a transitive continuous action of a Polish group G on a space X of second category in itself is necessarily 'open', or more accurately is microtransitive (the (continuous) evaluation map $e_x : g \mapsto g(x)$ takes open neighbourhoods E of 1_G to open neighbourhoods that are the orbit sets $E(x)$ of x). It emerges that this assertion is very close to the shift-compactness property: see [Ost2]. The Effros Theorem reduces to the Open Mapping Theorem when G, X are Banach spaces regarded as additive groups, and G acts on X by a linear surjection $L : G \rightarrow X$ via $g(x) = L(g) + x$. Indeed, here $e_0(E) = L(E)$. For a neat proof, choose an open neighbourhood U of 0 in G with $E \supseteq U - U$; then $L(U)$ is Baire (being analytic) and non-meagre (since $\{L(nU) : n \in \mathbb{N}\}$ covers X), and so $L(U) - L(U) \subseteq L(E)$ is an open neighbourhood of 0 in X .

C5. *Haar null and left Haar null.* The two families, which are both left and right translation-invariant (cf. [Sol2, p. 696] – if $\mu \in \mathcal{P}(G)$ witnesses that E is left Haar null, then $\mu_{g^{-1}}$ witnesses that Eg is left Haar null), coincide in Polish abelian groups, and in locally compact second countable groups (where they also coincide with the sets of Haar measure zero – by an application of the Fubini theorem). The former family, however, is in general smaller; indeed, non-Haar null sets need not have the Steinhaus-Weil property – see [Sol2].

C6. *Beyond local compactness: Haar category-measure duality.* In the absence of Haar measure, the definition of left Haar null subsets of a topological group G requires $\mathcal{U}(G)$, the universally measurable sets – by dint of the role of the totality of (probability) measures on G . The natural dual of $\mathcal{U}(G)$ is the class $\mathcal{U}_{\mathcal{B}}(G)$ of *universally Baire sets*, defined for G with a Baire topology as those sets B whose preimages $f^{-1}(B)$ are Baire in any compact Hausdorff space K for any continuous $f : K \rightarrow G$. Initially considered in [FenMW] for $G = \mathbb{R}$, these have attracted continued attention for their role in the investigation of axioms of determinacy and large cardinals – see especially [Woo]; cf. [MarS].

Analogously to the left Haar null sets, define a *left Haar meagre* set as any set M coverable by a universally Baire set B for which there are a compact Hausdorff space K and a continuous $f : K \rightarrow G$ with $f^{-1}(gB)$ meagre in K for all $g \in G$. These were introduced, in the abelian Polish group setting with K metrizable, by Darji [Dar], cf. [Jab], and shown there to form a σ -ideal of meagre sets (co-extensive with the meagre sets for G locally compact).

C7. *Non-separability.* The links between the Effros theorem in C5 above, the Baire theorem and the Steinhaus-Weil theorem are pursued at length in [Ost2]. There, any separability assumption is avoided. Instead *sequential* methods are used, for example shift-compactness arguments.

C8. *Metrizability and Christensen's Theorem.* An analytic topological group is metrizable; so if also it is a Baire space, then it is a Polish group – [HofT, Th. 2.3.6].

C9. *Strong Kemperman property: qualitative versus quantitative measure theory.* We note that property (*) of the Introduction and Lemma 1 corresponds to the following quantitative property on the line, stated in terms of Haar (i.e. Lebesgue) measure $\eta(\cdot) = |\cdot|$ for sets open in the Lebesgue density topology $\mathcal{D}_{\mathcal{L}}$:

(iv)* strong Kemperman property (see [Kem], [Kuc, Lemma 3.7.2]): for $0 \in$

$U \in \mathcal{D}_{\mathcal{L}}$ there is $\delta > 0$ so that for all $|t| < \delta$

$$|U \cap (t + U)| \geq \varepsilon.$$

This is connected with continuity of the Haar norm. Indeed, since

$$|U \cap (t + U)| = |U| - |U \Delta (t + U)|/2,$$

the inequality above is equivalent to

$$\|t\|_U^\eta := |U \Delta (t + U)| \leq 2(|U| - \varepsilon),$$

The latter holds for any $0 < \varepsilon < |U|$ and for sufficiently small t , by the continuity of the norm $\|t\|_U^\eta$.

On Fubini-Null and Measure upper-semicontinuity

Proof of Theorem FN. For μ -null $N \subseteq G$ the set $N \times G$ is $\mu \times \nu$ -null, so (by passing to the complement of the null exceptional set of the theorem) we may assume w.l.o.g. that the exceptional set of A is empty. By inner regularity, it suffices to show that $(\mu \times \nu)(K) = 0$ for all compact $K \subseteq A$.

For K compact, denote by F the (compact) projection of K on the first axis. Let $\varepsilon > 0$. By Prop. 5, for any $x \in F$ there is an open neighbourhood U_x of x and open V_x with $\nu(V_x) < \varepsilon$ and

$$K \cap (U_x \times G) \subseteq R_x := U_x \times V_x.$$

By compactness of F , there are $U^j \times V^j$ for $j = 1, \dots, n$, with U^j, V^j open and $\nu(V^j) < \varepsilon$ such that

$$F \subseteq \bigcup_j U^j : \quad K \subseteq \bigcup_j U^j \times V^j.$$

To disjoint the sets U^j , put

$$S^j := U^j \setminus \bigcup_{j < i} U^j : \quad \bigcup_j U^j = \bigcup_j S^j.$$

Then

$$F = \bigcup_j F \cap S^j : \quad K \subseteq \bigcup_j S^j \times V^j.$$

So

$$\mu(K) \leq \sum_j (\mu \times \nu)(S^j \times V^j) = \sum_j \mu(S^j) \nu(V^j) \leq \sum_j \mu(S^j) \cdot \varepsilon = \varepsilon \mu(F).$$

As $\varepsilon > 0$ was arbitrary, $\mu(K) = 0$. \square

Proof of Proposition 5 (Almost everywhere upper semicontinuity).

As this is much as in Prop. 2, suffice it to indicate the necessary adjustments. For measurable $E \subseteq G$ measurable and compact F with $\mu(F) > 0$, refer to

$$H := \Phi_E(F) = \bigcup_{x \in F} \{x\} \times xE.$$

Instead of the finite union of rectangles K , choose a compact set $K = K_\varepsilon \subseteq H$ with

$$(\mu \times \mu)(H \setminus K) < \varepsilon^2.$$

Then obtain a compact set $C = C_\varepsilon \subseteq F \setminus S_\varepsilon$ with $\mu(F \setminus C) < \varepsilon$ and

$$\mu(H_x \setminus K_x) \leq \varepsilon \quad (x \in C_\varepsilon).$$

For $x \in C_\varepsilon$, by upper semicontinuity of $t \mapsto \mu(K_t)$, there is a neighbourhood U_x^ε of x with

$$\mu(K_y) < \mu(K_x) + \varepsilon \quad (y \in U_x^\varepsilon \cap C).$$

So

$$\mu(H_y) < \mu(K_y) + \varepsilon < \mu(K_x) + 2\varepsilon < \mu(H_x) + 3\varepsilon,$$

i.e.

$$\mu(H_y) < \mu(H_x) + 3\varepsilon \quad (x \in C, y \in U_x^\varepsilon \cap C).$$

Thereafter the inductive argument of Prop. 4 follows almost verbatim. \square

On the Disaggregation Theorem

The sequence of Lemmas A-E below justifies the introduction of a new topology with sub-basic sets of the form $gB_\delta^{K,\Delta}$, but *only* on those points of G that can be *covered* by these sets: the detailed statement is in Theorem 9 below; Theorem 5, the Disaggregation Theorem of §4 is its immediate corollary. The proof strategy demands both a countable iteration – an inductive generation of a family of sets $B_\delta^{K,\Delta}$ – and then a countable subgroup of translators g . We put

$$\mathcal{B}_1 = \mathcal{B}_1(\mathbf{t}) := \{B_\delta^{K,\Delta}(\sigma(\mathbf{t})) : \gamma \in \Gamma, K \in \mathcal{H}, \delta, \Delta \in \mathbb{Q}_+, 0 < \sigma(K) < \Delta\}.$$

Lemma A. For $\mu \in \mathcal{P}(G)$, \mathbf{t} null and non-trivial, and arbitrary $\delta > 0$, if $0 < \Delta < \mu_-^{\mathbf{t}}(K)$, then $B_\delta^{K,\Delta}(\mu) \neq \{1_G\}$. In particular, for G amenable at 1_G and $\mu = \sigma(\mathbf{t})$, if $0 < \Delta < \sigma(K)$, then $B_\delta^{K,\Delta}(\sigma)$ is infinite.

Proof. Since \mathbf{t} is null and non-trivial, for all large enough n both $t_n \in B_\delta$ and also $\mu(Kt_n) > \Delta$. For $\mu = \sigma(\mathbf{t})$ and $0 < \Delta < \sigma(K)$, pick $0 < \theta < 1$ with

$$\theta\sigma(K) = \Delta.$$

Then for some, necessarily non-trivial, subsequence $\mathbf{s} := \{s_n\}$ of \mathbf{t} ,

$$\sigma(Ks_n) > \theta\sigma(K) = \Delta.$$

So $B_\varepsilon^{K,\Delta} = \{s \in B_\varepsilon : \sigma(Ks) > \Delta\}$ is infinite. \square

Lemma B. For $\sigma = \sigma(\mathbf{t})$ and $K \in \mathcal{K}(G) \cap \mathcal{M}_+(\sigma)$: if $w \in B_\delta^{K,\Delta}$, then for $H = Kw$ and some $\varepsilon > 0$

$$\{w\} \neq wB_\varepsilon^{H,\Delta} \subseteq B_\delta^{K,\Delta}, \text{ with } B_\varepsilon^{H,\Delta} \text{ infinite.}$$

In particular, if $1_G \in gB$ for some $B \in \mathcal{B}_1$, then there is $B' \in \mathcal{B}_1$ with $1_G \in B' \subseteq gB$.

Proof. As $w \in B_\delta$ there is $\varepsilon > 0$ with $wB_\varepsilon \subset B_\delta$. Then

$$\begin{aligned} w &\in wB_\varepsilon^{H,\Delta} = w\{s \in B_\varepsilon : \sigma(Hs) > \Delta\} = \{ws \in wB_\varepsilon : \sigma(Kws) > \Delta\} \\ &\subseteq \{x \in B_\delta : \sigma(Kx) > \Delta\} = B_\delta^{K,\Delta}; \end{aligned}$$

furthermore, $\sigma(H) = \sigma(Kw) > \Delta$, as $w \in B_\delta^{K,\Delta}$. The fact that $B_\varepsilon^{H,\Delta}$ is infinite follows from Lemma A, as $\sigma(H) = \sigma(Kw) > 0$, since $w \in B_\delta^{K,\Delta}$.

For the last part, suppose $1_G \in gB$ with $B = B_\delta^{K,\Delta} \in \mathcal{B}_1$; then $w \in B$ for $w = g^{-1}$. Applying the first part, take $B' := B_\varepsilon^{H,\Delta} \in \mathcal{B}_1$ for $H = Kw$ and the $\varepsilon > 0$ above; then,

$$w \in wB' = B_\varepsilon^{H,\Delta} \subseteq B_\delta^{K,\Delta} = B : \quad 1_G \in B' \subseteq gB. \quad \square$$

Corollary. If $x \in yB \cap zC$ for $x, y, z \in G$ and some $B, C \in \mathcal{B}_1$, then $x \in xB' \cap xC' \subseteq yB \cap zC$ for some $B', C' \in \mathcal{B}_1$.

Proof. As $1 \in x^{-1}yB$ and $1 \in x^{-1}zC$ there are $B', C' \in \mathcal{B}_1$ with $1 \in B' \subseteq x^{-1}yB$ and $1 \in C' \subseteq x^{-1}zC$. Then $x \in xB' \cap xC' \subseteq yB \cap zC$. \square

We now improve on Lemma B by including some technicalities, whose purpose is to introduce a *separable* topology refining that of G . As

$$B_d^D \subseteq B_\delta^\Delta \text{ for } 0 < d < \delta \text{ and } 0 < \Delta < D < \sigma(K),$$

we may restrict attention to $\delta, \Delta \in \mathbb{Q}_+ := \mathbb{Q} \cap (0, \infty)$.

Lemma C. *For $\sigma = \sigma(\mathbf{t})$ and countable $\mathcal{H} \subseteq \mathcal{K}(G) \cap \mathcal{M}_+(\sigma)$: there is a countable $D = D(\mathcal{H}) \subseteq G$ accumulating at 1_G such that: if $w \in B_\delta^{K,\Delta}$ with $K \in \mathcal{H}$, $\delta, \Delta \in \mathbb{Q}_+$ and $\Delta < \sigma(K)$, then for some $g \in D$ with $\sigma(Kg) > \Delta$ and some $\varepsilon \in \mathbb{Q}_+$,*

$$w \in gB_\varepsilon^{Kg,\Delta} \subseteq B_\delta^{K,\Delta}.$$

Proof. As G is separable, we may choose $\{\bar{g}_m\} = \{\bar{g}_m(B_\delta^{K,\Delta})\} \subseteq B_\delta^{K,\Delta}$ dense in $B_\delta^{K,\Delta}$, an infinite set, by Lemma A. Take

$$D = D(\mathcal{H}) := \{\bar{g}_m(B_\delta^{K,\Delta}) : K \in \mathcal{H}, \delta, \Delta \in \mathbb{Q}_+, \Delta < \sigma(K)\},$$

which is countable. Since $B_\delta^{K,\Delta} \subseteq B_\delta$, D accumulates at 1_G . We claim that D above satisfies the conclusions of the Lemma.

Fix $w \in B_\delta^{K,\Delta}$, with K, Δ, δ as in the hypotheses. Choose $\varepsilon \in \mathbb{Q}_+$ with

$$wB_{3\varepsilon} \subseteq B_\delta.$$

Choose $\bar{g}_m = \bar{g}_m(B_\delta^{K,\Delta})$ with $\|\bar{g}_m^{-1}w\| < \varepsilon$, possible by construction of $\{\bar{g}_m(B_\delta^{K,\Delta})\}$. Put $z_m := \bar{g}_m^{-1}w$; then $w = \bar{g}_m z_m$, $z_m \in B_\varepsilon$ and $\bar{g}_m \in wB_\varepsilon$, so $w \in \bar{g}_m z_m B_\varepsilon \subseteq \bar{g}_m B_{2\varepsilon} \subseteq wB_{3\varepsilon} \subseteq B_\delta$. By choice, $\sigma(K\bar{g}_m) > \Delta$, and furthermore

$$\begin{aligned} w &\in \bar{g}_m z_m \{s \in B_\varepsilon : \sigma(K\bar{g}_m z_m s) > \Delta\} \subseteq \bar{g}_m \{t \in B_{2\varepsilon} : \sigma(K\bar{g}_m t) > \Delta\} = \bar{g}_m B_{2\varepsilon}^{K\bar{g}_m,\Delta} \\ &\subseteq \{x \in B_\delta : \sigma(Kx) > \Delta\} = B_\delta^{K,\Delta}, \end{aligned}$$

as $\bar{g}_m B_{2\varepsilon} \subseteq B_\delta$. \square

In Lemma C above Kg need not belong to \mathcal{H} . Lemma D below asserts that Lemma C holds on a countable family \mathcal{H} of compact sets that is closed under the appropriate translation.

Lemma D. *There are a countable $\mathcal{H} \subseteq \mathcal{K}_+(\sigma)$ and a countable set $D = D(\mathcal{H}) \subseteq G$ dense in G such that if $w \in B_\delta^{K,\Delta}$ with $K \in \mathcal{H}$, $\delta, \Delta \in \mathbb{Q}_+$ and $0 < \sigma(K) < \Delta$, then for some $g \in D$ with $\sigma(Kg) > \Delta$ with $Kg \in \mathcal{H}$ and some $\varepsilon \in \mathbb{Q}_+$,*

$$w \in gB_\varepsilon^{Kg,\Delta} \subseteq B_\delta^{K,\Delta}.$$

Proof. Suppose σ is concentrated on $\bigcup_n K_n$, with each K_n compact. Taking \mathcal{H}_0 to comprise traces $K_n \cap g_m \bar{B}_\delta$ with $\{g_m\}$ dense in G and $\delta \in \mathbb{Q}_+$, proceed by induction:

$$\begin{aligned}\mathcal{H}_{n+1} &:= \{Kg : K \in \mathcal{H}_n, g \in D(\mathcal{H}_n), \delta, \Delta \in \mathbb{Q}_+, 0 < \sigma(K) < \Delta\}, \\ \mathcal{H} &:= \bigcup_n \mathcal{H}_n, \quad D := \bigcup_n D(\mathcal{H}_n). \quad \square\end{aligned}$$

Theorem 9. For $\sigma = \sigma(\mathbf{t})$ there are a countable $\mathcal{H} \subseteq \mathcal{K}_+(\sigma)$ and a countable set $\Gamma = \Gamma(\mathcal{H}) \subseteq G$ dense in G such that, taking

$$\mathcal{B}_{\mathcal{H}}(\mathbf{t}) = \{B_\delta^{K,\Delta}(\sigma) \in \mathcal{B}_1(\mathbf{t}) : K \in \mathcal{H}, \delta, \Delta \in \mathbb{Q}_+, 0 < \sigma(K) < \Delta\},$$

$$\mathcal{B}(\mathbf{t}) = \mathcal{B}_\Gamma(\mathbf{t}) := \Gamma \cdot \mathcal{B}_{\mathcal{H}}(\mathbf{t}) = \{\gamma B : \gamma \in \Gamma, B \in \mathcal{B}_{\mathcal{H}}(\mathbf{t})\}$$

is a sub-base for a second countable metrizable topology on the subspace

$$G(\sigma) := \bigcup \mathcal{B}_\Gamma(\mathbf{t}) = \bigcup \{\gamma B : B \in \mathcal{B}_{\mathcal{H}}(\mathbf{t}), \gamma \in \Gamma\}.$$

Proof. Take a countable subgroup Γ in G , which is dense in G under \mathcal{T}_G and contains $D(\mathcal{H})$, as in Lemma D. Consider $w \in \gamma B_\delta^{K,\Delta}$ with $\gamma \in \Gamma$, $K \in \mathcal{H}$, $\delta, \Delta \in \mathbb{Q}_+$, $\Delta < \sigma(K)$; then for some $g \in D(\mathcal{H}) \cap B_\delta^{K,\Delta}$ and $\varepsilon > 0$

$$gB_\varepsilon^{Kg,\Delta} \subseteq B_\delta^{K,\Delta},$$

and so both

$$w \in \gamma g B_\varepsilon^{Kg,\Delta} \subseteq \gamma B_\delta^{K,\Delta}$$

and $\gamma g \in \Gamma$. So, by the Corollary to Lemma B, the family $\mathcal{B}(\mathbf{t})$ forms a sub-base for a topology on the set of points

$$G(\sigma) := \bigcup \{\gamma B : B \in \mathcal{B}(\mathbf{t}), \gamma \in \Gamma\}.$$

For $K \in \mathcal{K}_+(\sigma)$ and any enumeration of \mathcal{H} as $K_n \in \mathcal{K}_+(\sigma)$, put for $x, y \in G$:

$$\rho_K(x, y) := |\mu(Kx) - \mu(Ky)|, \quad \rho(x, y) := \sup\{2^{-n} \rho_{K_n}(x, y)\}.$$

The latter metric generates the topology above on $G(\sigma)$. Indeed, for $K \in \mathcal{K}_+(\sigma)$, if $0 < \varepsilon < \mu(Kg)$

$$\begin{aligned}d_K(x, g) < \varepsilon &\implies [d(x, y) < \varepsilon \text{ and } \mu(Kg) - \varepsilon < \mu(Kx) < \mu(Kg) + \varepsilon], \\ [d(x, y) < \varepsilon \text{ and } \mu(Kg) - \varepsilon < \mu(Kx) < \mu(Kg) + \varepsilon] &\implies d_K(x, y) < 2\varepsilon.\end{aligned}$$

Write $x = gh$, then $d_L^G(x, g) < \varepsilon$ is equivalent to $\|h\| < \varepsilon$; as $x \mapsto \mu(Kx)$ is upper semicontinuous, there is $0 < \delta = \delta(\varepsilon) < \varepsilon$ such that $\mu(Kgh) < \mu(Kg) + \varepsilon$, for $h \in B_\delta$. So

$$B_\varepsilon^{d_K}(g) := \{x \in G : d_K(x, g) < \varepsilon\} \subseteq gB_\varepsilon^{Kg,\Delta}, \quad gB_{\delta(\varepsilon)}^{Kg,\Delta} \subseteq B_{2\varepsilon}^{d_K}(g),$$

for $\Delta = \mu(Kg) - \varepsilon$. \square

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